Random Matrices, Quantum Chaos
and Open Quantum Systems

Karol Życzkowski

in collaboration with

W. Bruzda, V. Cappellini, H.-J. Sommers, M. Smaczyński

Institute of Physics, Jagiellonian University, Cracow, Poland

and

Center for Theoretical Physics, Polish Academy of Sciences

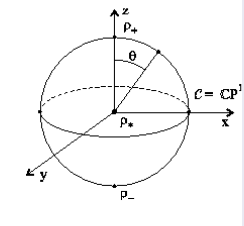
Maribor, Slovenia July 7, 2008

Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$

$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

Bloch sphere of $N = 2$ pure states



Space of pure states for an arbitrary N:

a complex projective space $\mathbb{C}P^{N-1}$ of $2N - 2$ real dimensions.

Unitary evolution

Fubini-Study distance in $\mathbb{C}P^{N-1}$

$$D_{FS}(|\psi\rangle, |\varphi\rangle) \;:=\; \arccos |\langle\psi|\varphi\rangle|$$

Unitary evolution

Let $U = \exp(iHt)$. Then $|\psi'\rangle = U|\psi\rangle$.
Since $|\langle\psi|\varphi\rangle|^2 = |\langle\psi|U^\dagger U|\varphi\rangle|^2$ any unitary evolution is a rotation in $\mathbb{C}P^{N-1}$

hence it is an isometry (with respect to any standard distance !)

Classical limit: what happend for large N?

How an isometry may lead to classically chaotic dynamics?
The limits $t \to \infty$ and $N \to \infty$ do not commute.

Closed systems, Unitary Dynamics & Quantum Chaos

'Quantum chaology': analogues of classically chaotic systems

Quantum analogues of classically chaotic dynamical systems can be described by random matrices

a). autonomous systems – Hamiltonians:

Gaussian ensembles of random Hermitian matrices, (GOE, GUE, GSE)

b). periodic systems – evolution operators:

Dyson circular ensembles of random unitary matrices, (COE, CUE, CSE)

Universality classes

Depending on the symmetry properties of the system one uses ensembles form orthogonal ($\beta = 1$); unitary ($\beta = 2$) and symplectic ($\beta = 4$) ensembles.

The exponent β determines the level repulsion, $P(s) \sim s^{\beta}$ for $s \to 0$ where s stands for the (normalised) level spacing, $s_i = \phi_{i+1} - \phi_i$.

see e.g. F. Haake, *Quantum Signatures of Chaos*

Set $\mathcal{M}_N$ of all mixed states of size N

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

example: $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - Bloch ball with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and convex:

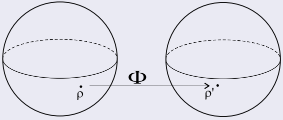
$\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i|$ where $a_i \geq 0$ and $\sum_i a_i = 1$.
It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.

How the set of all $N = 3$ mixed states looks like?

An 8 dimensional convex set with only 4 dimensional subset of pure (extremal) states, which belong to its 7–dim boundary

Quantum maps

Quantum operation: linear, completely positive trace preserving map



positivity: $\Phi(\rho) \geq 0, \quad \forall \rho \in \mathcal{M}_N$

complete positivity: $[\Phi \otimes \mathbb{1}_K](\sigma) \geq 0, \quad \forall \sigma \in \mathcal{M}_{KN}$ and $K = 2, 3, \ldots$

Enviromental form (open system !)

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes \omega_E)\;U^{\dagger}]\,.$$

where ω_E is an initial state of the environment while $UU^{\dagger} = \mathbb{1}$.

Kraus form

$\rho' = \Phi(\rho) = \sum_i A_i \rho A_i^{\dagger}$, where the Kraus operators satisfy

$\sum_i A_i^{\dagger} A_i = \mathbb{1}$, which implies that the trace is preserved.

Classical probabilistic dynamics & Markov chains

Stochastic matrices

Classical states: N-point probability distribution, $\mathbf{p} = \{p_1, \ldots p_N\}$,
where $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$
Discrete dynamics: $p_i' = S_{ij} p_j$, where S is a stochastic matrix of size N and maps the simplex of classical states into itself, $S: \ \Delta_{N-1} \to \Delta_{N-1}$.

Frobenius–Perron theorem

Let S be a stochastic matrix:

a) $S_{ij} \geq 0$ for $i, j = 1, \ldots, N$,

b) $\sum_{i=1}^{N} S_{ij} \ = \ 1$ for all $j = 1, \ldots, N$.

Then

i) the spectrum $\{z_i\}_{i=1}^{N}$ of S belongs to the unit disk,

ii) the leading eigenvalue equals unity, $z_1 = 1$,

iii) the corresponding eigenstate $\mathbf{p}_{\rm inv}$ is invariant, $S\mathbf{p}_{\rm inv} = \mathbf{p}_{\rm inv}$.

Quantum stochastic maps (trace preserving, CP maps)

Superoperator $\Phi: \quad \mathcal{M}_N \to \mathcal{M}_N$

A *quantum operation* can be described by a matrix Φ of size N^2,

$$\rho' = \Phi\rho \qquad \text{or} \qquad \rho'_{m\mu} = \Phi_{\substack{m\mu \\ n\nu}} \rho_{n\nu} \ .$$

The superoperator Φ can be expressed in terms of the Kraus operators A_i,
$\Phi = \sum_i A_i \otimes \bar{A}_i$.

Dynamical Matrix D: Sudarshan et al. (1961)

obtained by *reshuffling* of a 4–index matrix Φ is Hermitian,

$$D_{\substack{mn \\ \mu\nu}} := \Phi_{\substack{m\mu \\ n\nu}} \ , \quad \text{so that} \quad D_\Phi = D_\Phi^\dagger =: \Phi^R \ .$$

Theorem of Choi (1975). A map Φ is completely positive (CP) if and only if the dynamical matrix D is positive, $D \geq 0$.

Spectral properties of a superoperator Φ

Quantum analogue of the Frobenious-Perron theorem

Let Φ represent a stochastic quantum map, i.e.

a') $\Phi^R \geq 0$; (Choi theorem)

b') $\mathrm{Tr}_A \Phi^R = \mathbb{1} \Leftrightarrow \sum_k \Phi_{\substack{kk\\ij}} = \delta_{ij}$. (trace preserving condition)

Then

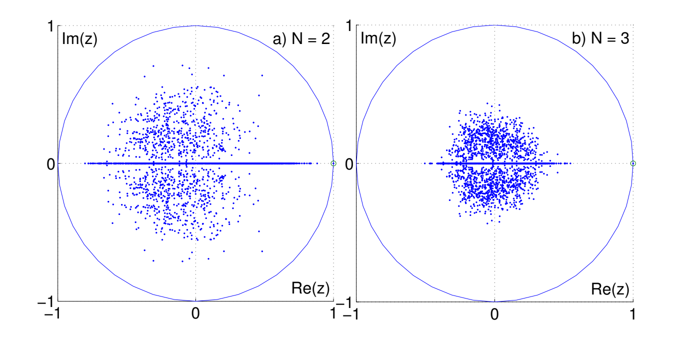
i') the spectrum $\{z_i\}_{i=1}^{N^2}$ of Φ belongs to the unit disk,

ii') the leading eigenvalue equals unity, $z_1 = 1$,

iii') the corresponding eigenstate (with N^2 components) forms a matrix ω of size N, which is positive, $\omega \geq 0$, normalized, $\mathrm{Tr}\omega = 1$, and is invariant under the action of the map, $\Phi(\omega) = \omega$.

Classical case

In the case of a diagonal dynamical matrix, $D_{ij} = d_i \delta_{ij}$ reshaping its diagonal $\{d_i\}$ of length N^2 one obtains a matrix of size N, where $S_{ij} = D_{\substack{ii\\jj}}$, of size N which is stochastic and recovers the standard F–P theorem.



Spectra of several random superoperators Φ for a) $N = 2$ and b) $N = 3$ contain:

i) the leading eigenvalue $z_1 = 1$ corresponding to the invariant state ω,

ii) real eigenvalues,

iii) complex eigenvalues inside the disk of radius $r = |z_2| \leq 1$.

Random (classical) stochastic matrices

Ginibre ensemble of complex matrices

Square matrix of size N, all elements of which are independent random complex Gaussian variables.

An algorithm to generate S at random:

1) take a matrix X form the complex Ginibre ensemble
2) define the matrix S,

$$S_{ij} \; := \; |X_{ij}|^2 / \sum_{i=1}^{N} |X_{ij}|^2 \; ,$$

which is stochastic by construction:

each of its columns forms an independent random vector distributed uniformly in the probability simplex Δ_{N-1}.

Random quantum states

How to generate a mixed quantum state at random?

1) Fix $M \geq 1$ and take a $N \times M$ random complex Ginibre matrix X;
2) Write down the positive matrix $Y := XX^\dagger$,
3) Renormalize it to get a random state ρ,

$$\rho \ := \ \frac{Y}{\mathrm{Tr}\, Y} \ .$$

This matrix is positive, $\rho \geq 0$ and normalised, $\mathrm{Tr}\rho = 1$, so it represents a quantum state!

Special case of $M = N$ (square Ginibre matrices)

Then random states are distributed uniformly with respect to the Hilbert-Schmidt (flat) measure,
e.g. for $M = N = 2$ random mixed states cover uniformly the interior of the Bloch ball.

Random (quantum) stochastic maps

An algorithm to generate Φ at random:

1) Fix $M \geq 1$ and take a $N^2 \times M$ random complex Ginibre matrix X;
2) Find the positive matrix $Y := \mathrm{Tr}_A XX^\dagger$ and its square root $\sqrt{Y}$;
3) Write the dynamical matrix (*Choi matrix*)

$$D = (\mathbb{1}_N \otimes \frac{1}{\sqrt{Y}}) XX^\dagger (\mathbb{1}_N \otimes \frac{1}{\sqrt{Y}}) \ ;$$

4) Reshuffle the Choi matrix to obtain the superoperator $\Phi = D^R$ and use to produce a random map, $\rho'_{m\mu} = \Phi_{\substack{m\mu \\ n\nu}} \rho_{n\nu}$.

Map Φ obtained in this way is stochastic !
i.e. Φ is completely positive and trace preserving

Random stochastic maps II

Probability distribution for random maps

$$P(D) \propto \det(D^{M-N^2})\,\delta(\mathrm{Tr}_A D - \mathbb{1}) ,$$

In the special case $M = N^2$ the determinant vanishes, so there are other constraints on the distribution of the random Choi matrix D, besides the partial trace relation, $\mathrm{Tr}_A D = \mathbb{1}$.

Interaction with M–dim. enviroment

1') Chose a random unitary matrix U
according to the Haar measure on $U(NM)$

2') Construct a random map defining

$$\rho' = \mathrm{Tr}_M[U(\rho \otimes |\nu\rangle\langle\nu|)U^\dagger] ,$$

where $|\nu\rangle \in \mathcal{H}_M$ is an arbitrary (fixed) state of the environment.

Bloch vector representation of any state ρ

$$\rho = \sum_{i=0}^{N^2-1} \tau_i \, \lambda^i$$

where λ^i are generators of SU(N) such that $\mathrm{tr}\left(\lambda^i \lambda^j\right) = \delta^{ij}$ and $\lambda^0 = \mathbb{1}/\sqrt{N}$ and τ_i are expansion coefficients.
Since $\rho = \rho^\dagger$, the generalized Bloch vector $\vec{\tau} = [\tau_0, \ldots, \tau_{N^2-1}]$ is real.

Stochastic map Φ in the Bloch representation

The action of the map Φ can be represented as

$$\tau' = \Phi(\tau) = C\tau + \kappa,$$

where C is a real, asymmetric contraction matrix of size $N^2 - 1$ while κ is a translation vector. Thus $\Phi = \begin{bmatrix} 1 & 0 \\ \kappa & C \end{bmatrix}$ and the eigenvalues of C are also eigenvalues of Φ.

Random maps & random matrices

Full rank, symmetric case $M = N^2$

For large N the measure for C can be described by the real Ginibre ensemble of non-hermitian Gaussian matrices.

Spectral density in the unit disk

The spectrum of Φ consists of:

i) the leading eigenvalue $z_1 = 1$,

ii) the component at the real axis, the distribution of which is asymptotically given by the step function $P(x) = \frac{1}{2}\Theta(x-1)\Theta(1-x)$,

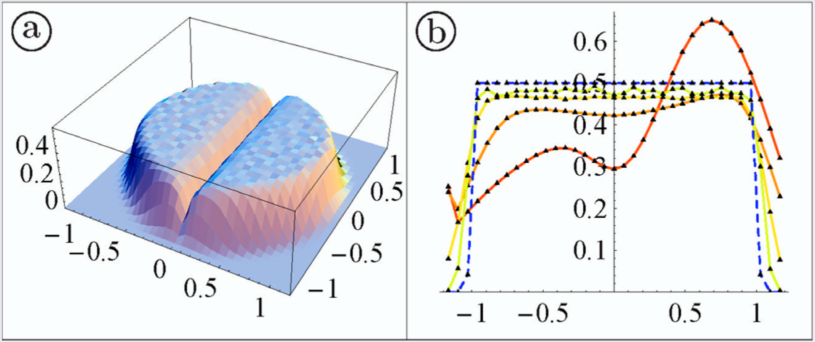
iii) complex eigenvalues, which cover the disk of radius $r = |z_2| \leq 1$ uniformly according to the Girko distribution.

Subleading eigenvalue $r = |z_2|$

The radius r is determined by the trace condition: Since the avearge $\langle \mathrm{Tr} D^2 \rangle = \langle \mathrm{Tr} \Phi\Phi^\dagger \rangle \approx \mathrm{const}$ then $r \sim 1/N$ so the spectrum of the rescaled matrix $\Phi' := N\Phi$ covers the entire unit disk.

Spectral density for random stochastic maps

Numerical results



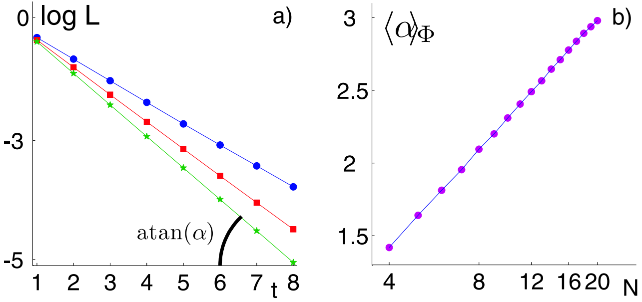
a) Distribution of complex eigenvalues of 10^4 rescaled random operators $\Phi' = N\Phi$ already for $N = 10$ can be approximated by the circle law of Girko.

b) Distribution of real eigenvalues $P(x)$ of Φ' plotted for $N = 2, 3, 7$ and 14 tends to the step function!

Convergence to equilibrium and decoherence rate

Average trace distance to invariant state $\omega = \Phi(\omega)$

$L(t) = \langle \mathrm{Tr}|\Phi^t(\rho_0) - \omega| \rangle_\psi$, where the average is performed over an ensemble of initially pure random states, $\rho_0 = |\psi\rangle\langle\psi|$.
Numerical result confirm an exponential convergence, $L(t) \sim \exp(-\alpha t)$.



a) Average trace distance of random pure states to the invariant state of Φ as a function of time for $N = 4(\bullet)$, $6(\blacksquare)$, $8(\star)$.
b) mean convergence rate $\langle\alpha\rangle_\Phi$ scales as $\ln N$ with the system size N.

Comparison with a quantum dynamical system

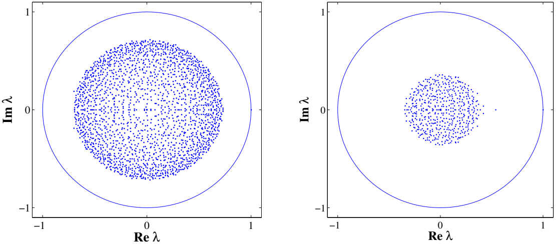
Generalized quantum baker map with measurements

a) Standard quantisation of Balazs and Voros $B = F_N^{\dagger} \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix}$,

where F_N denotes the Fourier matrix of size N. Then $\rho' = B\rho_i B^{\dagger}$

b) M measurement operators projecting into orthogonal subspaces

$\rho_{i+1} = \sum_{i=1}^{M} P_i \rho' P_i$



Numerical spectra of superoperator of baker map for $N = 64$, a) $M = 2$,
b) $M = 8$ (*master thesis of M. Smaczyński, in preparation*).

Concluding Remarks

- Quantum Chaos:
  a) in case of closed systems one studies unitary evolution operators and characterizes their spectral properties,
  b) for open, interacting systems one analyzes non–unitary time evolution described by quantum stochastic maps.
- We analyzed spectral properties of quantum stochastic maps and formulated a quantum analogue of the Frobenius-Perron theorem.
- A natural flat measure in the space of quantum operations (stochastic maps) is defined and an algorithm to produce them at random is given.
- For large N random quantum operations can be described by random matrices of the real Ginibre ensemble.
- Sequential action of a fixed random map brings all pure states to the invariant state exponentially fast. The convergence rate scales logarithmically with the system size N.