Microwave studies of chaotic systems

Lecture 2: The poles of the scattering matrix

Hans-Jürgen Stöckmann

`stoeckmann@physik.uni-marburg.de`

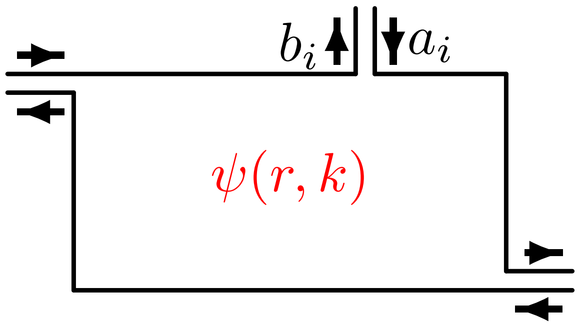
Fachbereich Physik, Philipps-Universität Marburg, Germany

Contents

- Billiards as scattering systems
- The isolated resonance regime
- Reflection fluctuations
- The Harmonic Inversion
- Line width distributions
- Summary

Billiards as scattering systems

Green function approach

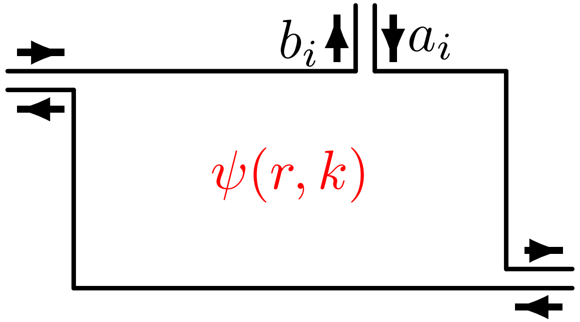


microwave billiard with N attached wave-guides (Stöckmann *et al.* 2002).

$\psi(r,k)$: wave function within the billiard

$\bar{G}(r,r',k) = \sum_n \frac{\bar{\psi}_n(r)\bar{\psi}_n(r')}{k^2-\bar{k}_n^2}$: billiard Green function with mixed boundary conditions ($\psi = 0$ on the boundary, $\nabla\psi = 0$ at the waveguides).

Green function approach



microwave billiard with N attached wave-guides (Stöckmann *et al.* 2002).

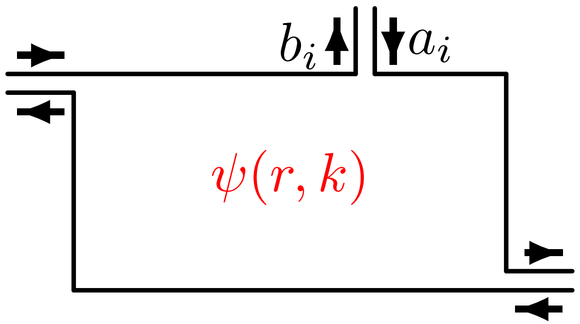
$\psi(r,k)$: wave function within the billiard

$\bar{G}(r,r',k) = \sum_n \frac{\bar{\psi}_n(r)\bar{\psi}_n(r')}{k^2-\bar{k}_n^2}$: billiard Green function with mixed boundary conditions ($\psi = 0$ on the boundary, $\nabla\psi = 0$ at the waveguides).

Both $\psi(r,k)$ and $\bar{G}(r,r',k)$ obey Helmholtz equations:

$$\begin{aligned} \left(\Delta + k^2\right)\psi(r,k) &= 0 \\ \left(\Delta + k^2\right)\bar{G}(r,r',k) &= \delta(r-r') \end{aligned}$$

Green function approach



microwave billiard with N attached wave-guides (Stöckmann *et al.* 2002).

$\psi(r,k)$: wave function within the billiard

$\bar{G}(r,r',k) = \sum_n \frac{\bar{\psi}_n(r)\bar{\psi}_n(r')}{k^2-\bar{k}_n^2}$: billiard Green function with mixed boundary conditions ($\psi = 0$ on the boundary, $\nabla\psi = 0$ at the waveguides).

Both $\psi(r,k)$ and $\bar{G}(r,r',k)$ obey Helmholtz equations:

$$\begin{aligned} \left(\Delta + k^2\right)\psi(r,k) &= 0 & &|\times\bar{G}(r,r',k) \\ \left(\Delta + k^2\right)\bar{G}(r,r',k) &= \delta(r-r') & &|\times\psi(r,k) \end{aligned}$$

$$\psi(r,k)\Delta\bar{G}(r,r',k) - \bar{G}(r,r',k)\Delta\psi(r,k) = \psi(r,k)\delta(r-r')$$

Green function approach (*cont.*)

Integrating over the billiard area and applying Green's theorem we obtain

$$\begin{aligned}\psi(r',k) &= \int dS\left[\psi(r,k)\nabla_\perp\bar{G}(r,r',k)-\bar{G}(r,r',k)\nabla_\perp\psi(r,k)\right]\\ &= -L\sum_i \bar{G}(r_i,r',k)\nabla_\perp\psi(r_i,k)\end{aligned}$$

L: width of the waveguides

Green function approach (*cont.*)

Integrating over the billiard area and applying Green's theorem we obtain

$$\begin{aligned}\psi(r',k) &= \int dS\left[\psi(r,k)\nabla_\perp \bar{G}(r,r',k) - \bar{G}(r,r',k)\nabla_\perp \psi(r,k)\right] \\ &= -L\sum_i \bar{G}(r_i,r',k)\nabla_\perp \psi(r_i,k)\end{aligned}$$

L: width of the waveguides

In the guides we have

$$\psi(r,k) = a_i e^{\imath k|r-r_i|} - b_i e^{-\imath k|r-r_i|}$$

The solutions are matched at the coupling positions:

$$a_i - b_i = \imath k L\sum_j \bar{G}_{ij}(a_j+b_j)\,, \qquad \bar{G}_{ij} = \bar{G}(r_i,r_j,k)$$

The scattering matrix

In matrix short-hand notation $\Longrightarrow$

$$a - b = \imath kL\bar{G}(a+b)\,, \qquad \bar{G} = W^\dagger \tfrac{1}{E-H} W$$

Comparison with definition $b = Sa$ of the scattering matrix yields

$$S = \frac{1 - \imath\gamma\bar{G}}{1 + \imath\gamma\bar{G}}\,, \qquad \gamma = kL$$

$\Longrightarrow$ A measurement of S thus yields the modified Green function $\bar{G}$!

The scattering matrix

In matrix short-hand notation $\Longrightarrow$

$$a - b = \imath k L \bar{G}(a+b)\,, \qquad \bar{G} = W^\dagger \tfrac{1}{E-H} W$$

Comparison with definition $b = Sa$ of the scattering matrix yields

$$S = \frac{1 - \imath\gamma\bar{G}}{1 + \imath\gamma\bar{G}}\,, \qquad \gamma = kL$$

$\Longrightarrow$ A measurement of S thus yields the modified Green function $\bar{G}$!

This is equivalent to

$$S = 1 - 2\imath W^\dagger \frac{1}{E - H_{\rm eff}} W$$

where W is the matrix with elements $W_{nm} = \sqrt{\gamma}\bar{\psi}_n(r_m)$, and

$$H_{\rm eff} = \bar{H} - \imath W W^\dagger$$

The isolated resonance regime

The billiard Breit-Wigner formula

For isolated resonances

$$S = 1 - 2\imath W^\dagger \frac{1}{E - H_{\mathrm{eff}}} W$$

reduces to

$$S_{ij}(E) = \delta_{ij} - 2\imath\gamma \sum_n \frac{\bar\psi_n(r_i)\bar\psi_n(r_j)}{E - \bar E_n + \frac{\imath}{2}\Gamma_n}\,, \qquad \Gamma_n = 2\gamma\sum_k \left|\bar\psi_n(r_k)\right|^2$$

A transmission measurement as a function of antenna positions thus yields the modified Green function.

The billiard Breit-Wigner formula

For isolated resonances

$$S = 1 - 2\imath W^\dagger \frac{1}{E-H_{\text{eff}}} W$$

reduces to

$$S_{ij}(E) = \delta_{ij} - 2\imath\gamma \sum_n \frac{\bar{\psi}_n(r_i)\bar{\psi}_n(r_j)}{E - \bar{E}_n + \frac{\imath}{2}\Gamma_n}\,, \qquad \Gamma_n = 2\gamma \sum_k \left|\bar{\psi}_n(r_k)\right|^2$$

A transmission measurement as a function of antenna positions thus yields the modified Green function.

Random matrix assumption: $\psi_n(r)$ is Gaussian distributed!

Allows calculation of distribution of

- Resonance depths $|\psi_n(r)|^2$
- Line widths $\Gamma_n = 2\gamma \sum_k |\psi_n(r_k)|^2$

The Porter-Thomas distribution

Line width distribution function:

$$P_\nu(x) = \left\langle \delta \left(x - \sum_{k=1}^{\nu} \left| \psi_n \left(r_k \right) \right|^2 \right) \right\rangle$$

$$= \left(\tfrac{A}{2\pi}\right)^{\frac{\nu}{2}} \int \delta \left(x - \sum_{k=1}^{\nu} \left| \psi_k \right|^2 \right) \prod_k \exp\left(-\tfrac{A}{2}\psi_k^2\right)\, dk$$

Introducing ν-dimensional polar coordinates the integration is trivial and yields χ^2 distribution

$$P_\nu(x) = \left(\tfrac{A}{2}\right)^{\frac{\nu}{2}} \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} \exp\left(-\tfrac{A}{2}x\right)$$

The Porter-Thomas distribution

Line width distribution function:

$$P_\nu(x) = \left\langle \delta\left(x - \sum_{k=1}^{\nu} \left|\psi_n\left(r_k\right)\right|^2\right)\right\rangle$$

$$= \left(\frac{A}{2\pi}\right)^{\frac{\nu}{2}} \int \delta\left(x - \sum_{k=1}^{\nu} \left|\psi_k\right|^2\right) \prod_k \exp\left(-\frac{A}{2}\psi_k^2\right)\, dk$$

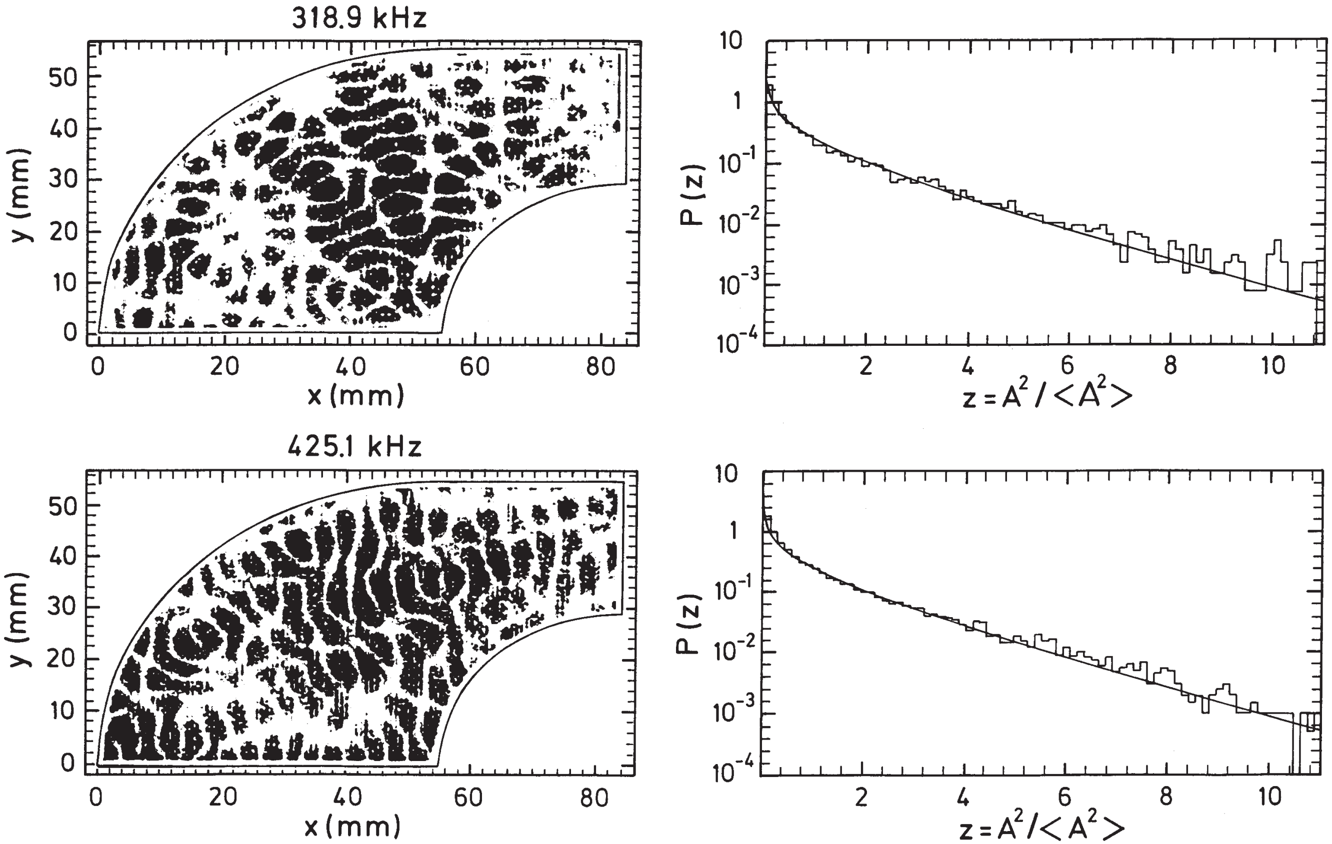
Introducing ν-dimensional polar coordinates the integration is trivial and yields χ^2 distribution

$$P_\nu(x) = \left(\frac{A}{2}\right)^{\frac{\nu}{2}} \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} \exp\left(-\frac{A}{2}x\right)$$

For distribution of resonance depths one gets in particular a Porter-Thomas distribution

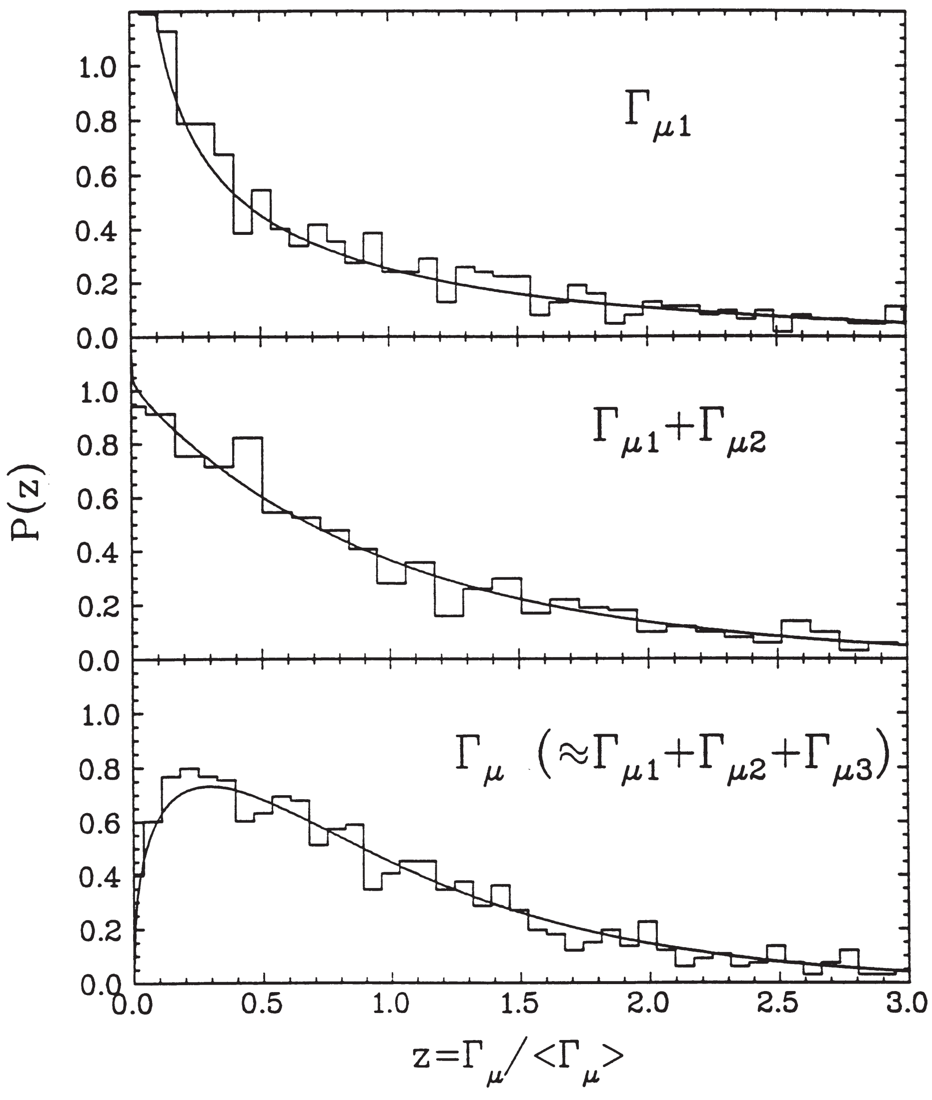
$$P_1(x) = \sqrt{\frac{A}{2\pi x}} \exp\left(-\frac{A}{2}x\right)$$

Vibrating plates



Porter-Thomas distributions in the squared amplitude distribution functions of vibrating silicon plates of a quarter stadium-Sinai billiard (K. Schaadt, diploma work, Copenhagen 1997).

χ^2 distributions in microwave billiards

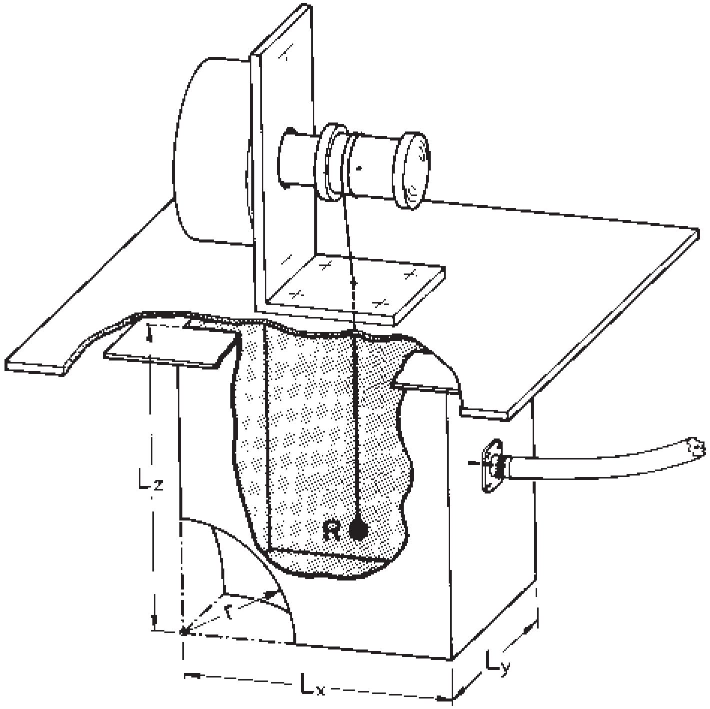


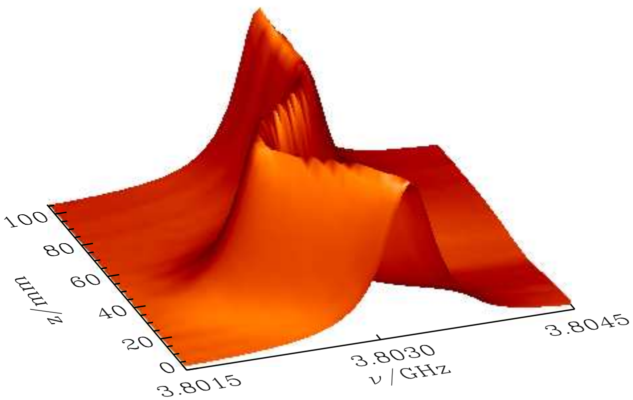
Distribution of partial widths for a superconducting quarter stadium billiard with three attached antennas (Alt *et al.* 1995).

top: one antenna
middle: two antennas
bottom: linewidth

Solid lines:
χ^2 distributions with $\nu = 1, 2, 3$.

Three-dimensional billiards





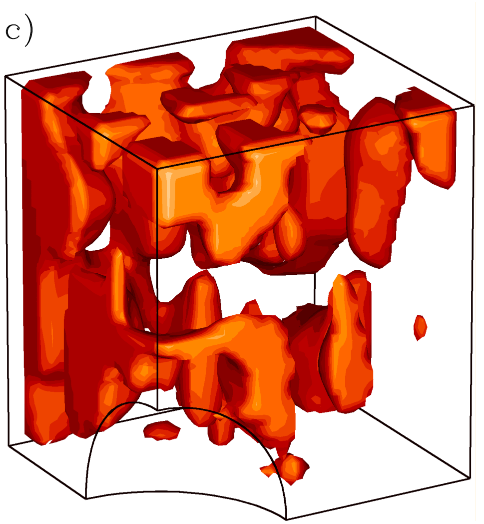
Perturbing bead method measures frequency shift proportional to

$$\Delta\nu \sim -2\mathbf{E}^2 + \mathbf{B}^2$$

$\mathbf{E}$,$\mathbf{B}$: electromagnetic fields at the perturber position

Three-dimensional billiards (*cont.*)

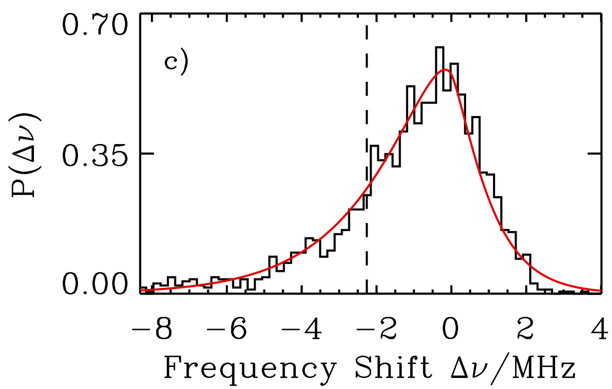
Assuming that all six field components are uncorrelated, the distribution function of frequency shifts is given by a generalized χ^2 distribution (Dörr *et al.* 1998)



$$P(\Delta\nu) = \frac{\sqrt{2}\alpha^2}{3\pi}|\Delta\nu| \exp\left(-\alpha\frac{\Delta\nu}{4}\right) K_1\left(\frac{3}{4}\alpha|\Delta\nu|\right)$$

Top: typical field distribution

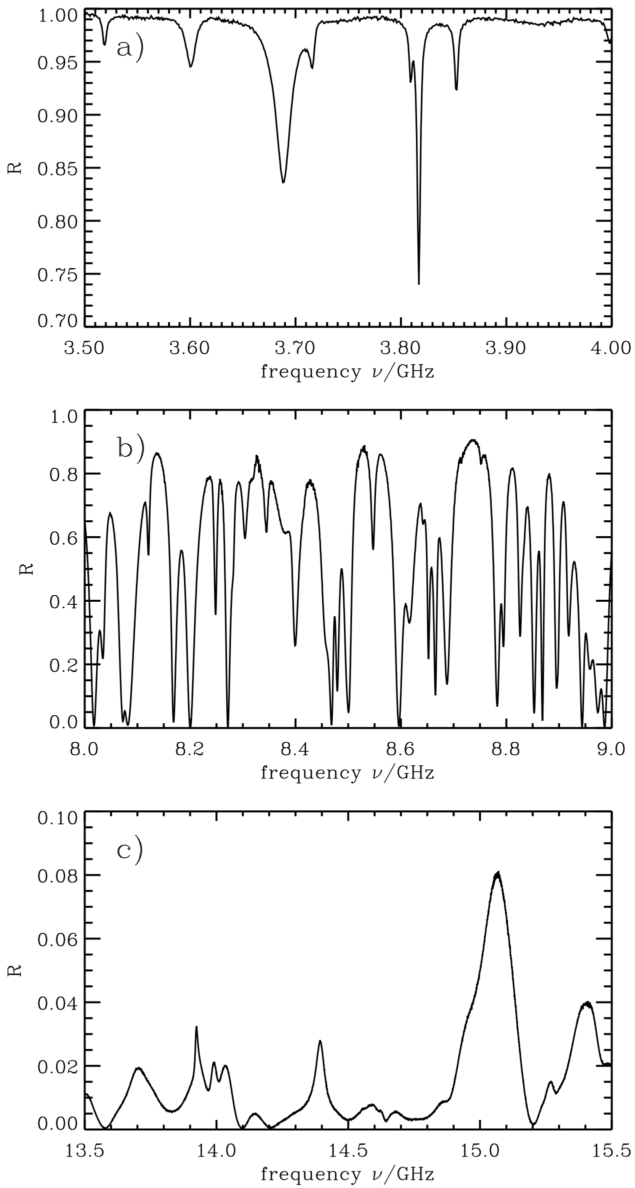
Bottom: corresponding frequency shift distribution $P(\Delta\nu)$.

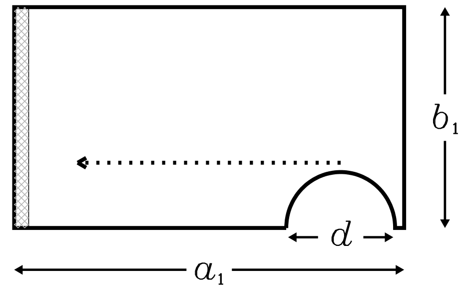


$\Longrightarrow$ For chaotic field distributions the fields can be considered as a random superposition of plane waves !

Reflection fluctuations

Influence of absorption



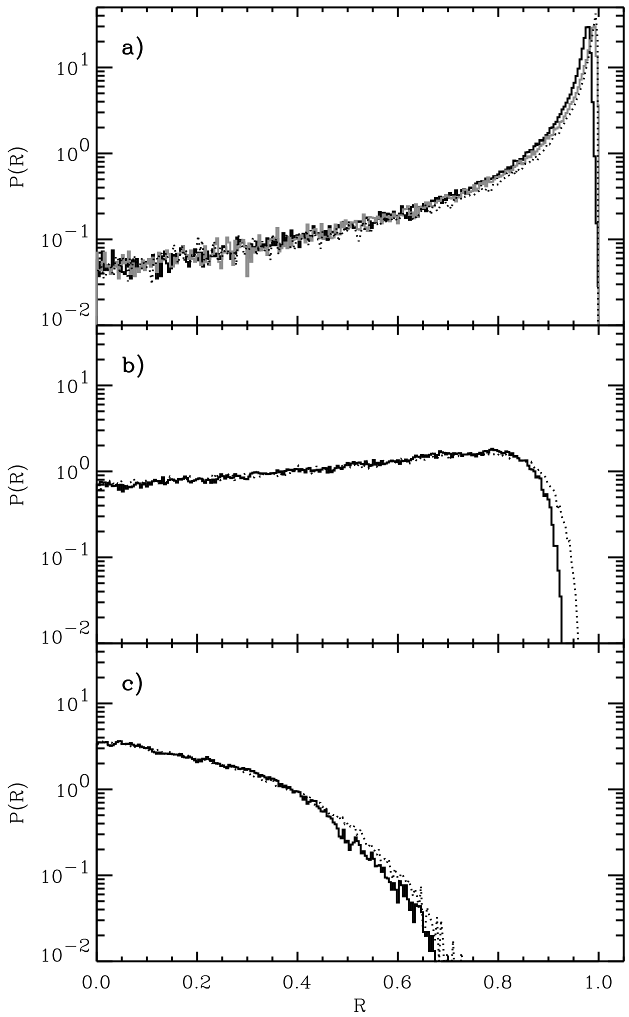


How does the distribution of reflection coefficients R vary with increasing absorption?

(R. Mendez *et al.* 2003)

Analytical results in the limits of weak absorption (Beenakker, Brouwer 2001) and strong absorption (Kogan *et al.* 2000).

Distribution of reflection coefficients



— : experiment
— : simulation

Simulation parameters (antenna coupling, wall absorption) taken from the experiment.

$\Longrightarrow$ There are no free parameters!

Results confirmed by theory (Fyodorov, Savin 2004).

The Harmonic Inversion

Problem

- Extremely difficult to resolve resonances in the regime of strong overlap
- Therefore up to now only results on average properties such as distribution of transmission coefficient etc. available

Alternative: Harmonic Inversion

- Essential developments by
  - Wall, Neuhauser 1995
  - Mandelshtam, Taylor 1997
- Brought into a manageable form by
  - Main 1999

The following presentation follows the paper by Wiersig, Main 2007

Technique

- Exponentially decaying time signal

$$c(t) = \sum_{k=1}^{K} d_k e^{-\imath \omega_k t}, \qquad \omega_k = \Omega_k - \tfrac{\imath}{2}\Gamma_k$$

- Discretization

$$c_n = c(n\tau) = \sum_{k=1}^{K} d_k \left(z_k\right)^n, \qquad z_k = e^{-\imath \omega_k \tau}$$

- Discretized Mellin transform

$$g(z) = \sum_{n=0}^{\infty} c_n z^{-n} = \sum_{k=1}^{K} d_k \sum_{n=0}^{\infty} \left(\tfrac{z_k}{z}\right)^n$$

- Summation of geometric series

$$g(z) = \sum_{k=1}^{K} \frac{z d_k}{z - z_k} = \frac{P_K(z)}{Q_K(z)}$$

$P_K(z), Q_K(z)$: Polynomials of degree K

Technique (*cont.*)

$$g(z) = \sum_{k=1}^{K} \frac{z d_k}{z - z_k} = \frac{P_K(z)}{Q_K(z)}$$

$z_k = e^{-\imath\omega\tau}$: Zeros of $Q_K(z)$

$$d_k = \frac{P_K(z_k)}{z_k Q'_K(z_k)}$$

Now the crucial point:

Knowledge of $2K$ signalpoints $c_0, \ldots, c_{2K-1}$ is sufficient to calculate the coefficients of the two polynomials

$$P_K(z) = \sum_{k=1}^{K} b_k z^k \,, \qquad Q_K(z) = \sum_{k=1}^{K} a_k z^k - 1$$

Technique (*cont.*)

- Coefficients a_k of $Q_K(z)$ obtained as solutions of the linear set of equations

$$c_n = \sum_{k=1}^{K} c_{n+k} a_k \,, \qquad n = 0, \ldots, K-1$$

- Once the a_k are known, the coefficients b_k of $P_K(z)$ are obtained from

$$b_k = \sum_{m=0}^{K-k} a_{k+m} c_m \,, \qquad k = 1, \ldots, K$$

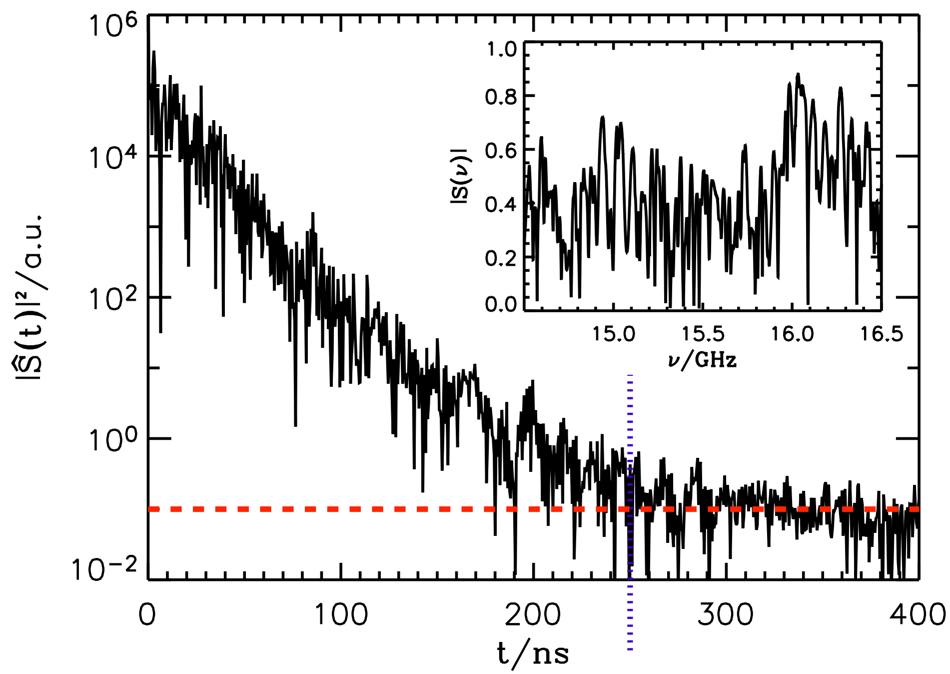
- With $Q_K(z)$ and $P_K(z)$ known, the ω_k and d_k can be determined as described above.

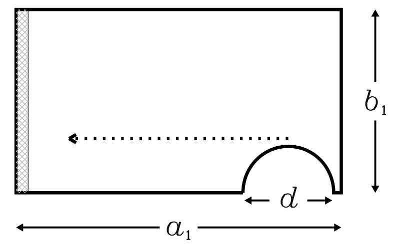
Points to be considered

- Conditions
  - Complex time signal needed
  - Time signal must be a superposition of damped exponentials
  - Number of data points must exceed the number of resonances by a factor of 2
- Virtues of the technique
  - No fit necessary
  - Number of resonances may be unknown
- Obvious problem
  - How to become rid of spurious resonances?
- Question
  - Is the technique sufficiently robust to cope with experimental data?

Line width distributions

Fourier transform of the spectrum



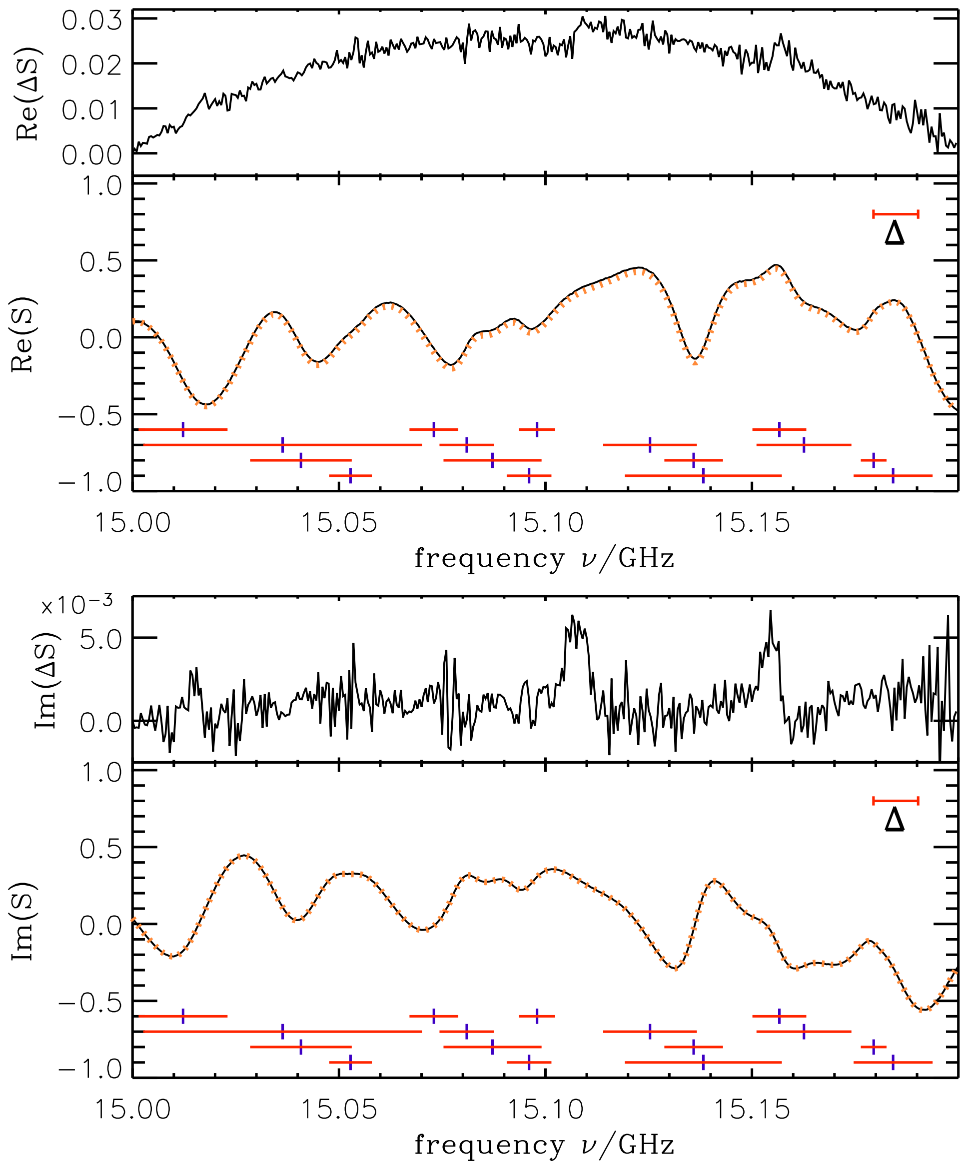


$$S(\nu) = 1 - \sum_n \frac{a_n}{\nu - \nu_n + i\gamma_n}$$

$\Longrightarrow$

$$\hat{S}(t) = \begin{cases} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-2\pi i \nu t} S(\nu) \mathrm{d}\nu = \delta(t) - \sum_n a_n e^{-2\pi i(\nu_n - i\gamma_n)t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Harmonic inversion



Real (top) and imaginary (bottom) part of the spectrum

solid line: original spectrum

dotted line: reconstructed spectrum

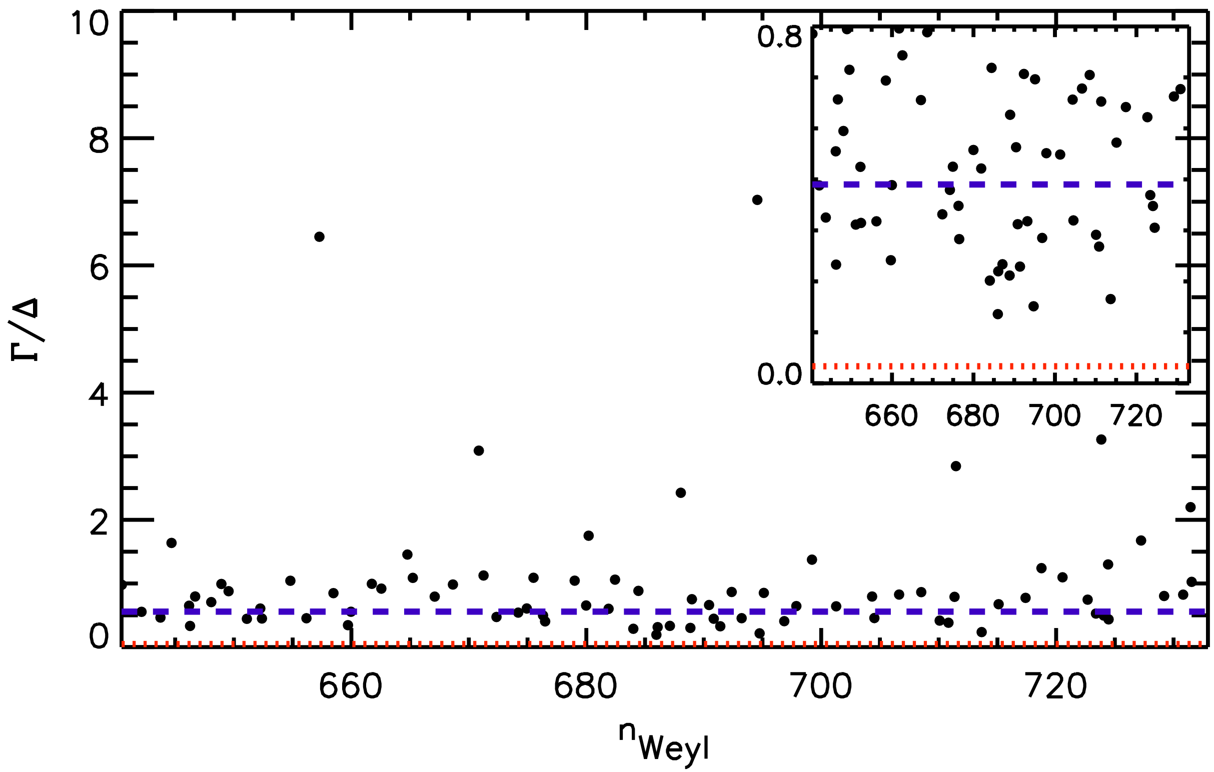
on top: difference

horizontal and vertical bars: positions and widths of the found resonances

Found resonances: 16

Expected (Weyl formula): 18

Pole distribution in the complex plane



red: expected from wall absorption (skin effect)

blue: expected from overall exponential decay (Schäfer *et al.* 2003),

$$\hat{S}(t) \sim e^{-\lambda t}$$

Line widths distribution

Sommers *et al.* 1999: Exact results for arbitrary number of channels

However: "Rather awkward even for the simplest case!"

One channel case:

$$P(y) = \frac{1}{4}\frac{\partial^2}{\partial y^2}\int_{-1}^{1} \mathrm{d}\lambda\ (1-\lambda^2)e^{2\pi\lambda y}F(\lambda, y)\,, \qquad y = \Gamma/\Delta$$

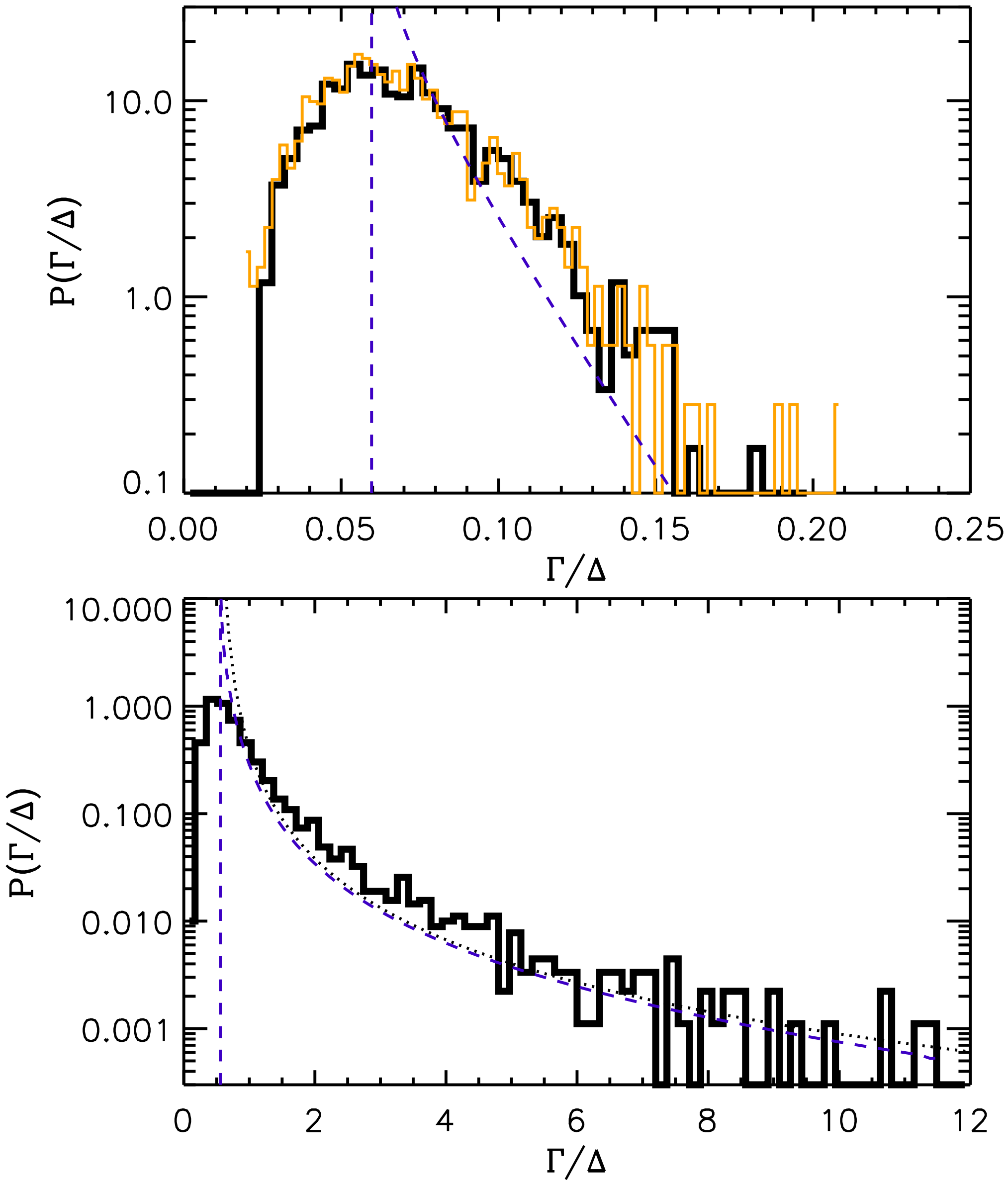
where

$$F(\lambda, y) = (g-\lambda)\int_g^\infty \mathrm{d}p_1 \frac{e^{-\pi y p_1}}{(\lambda-p_1)^2\sqrt{(p_1^2-1)(p_1-g)}} \int_1^g \mathrm{d}p_2 \frac{(p_1-p_2)e^{-\pi y p_2}}{(\lambda-p_2)^2\sqrt{(p_2^2-1)(g-p_2)}}$$

$$g = \frac{2}{T_a} - 1$$

For g=1 (perfect coupling): $P(y) \sim 1/(4\pi y^2)$

Experimental results



—: from Harmonic Inversion

—: from ordinary fit

- - -: theory

Assumption:

Line width due to one-channel coupling (antenna) and constant wall absorption.

Quite good agreement for high frequencies (bottom), but for low frequencies (top) there is something missing!

Interpretation

There are additional channels!

The exact formulas (Sommers *et al.* 1999) posed tremendous numerical problems.

Therefore a phenomenological approach has been used for the line width distribution:

$$p(\Gamma) = \int \chi_\nu(\Gamma - \hat{\Gamma}) p_0(\Gamma)\, d\,\hat{\Gamma}$$

$p_0(\Gamma)$: one-channel distribution (antenna)

$\chi^2_\nu(\Gamma)$: chi-square distribution with ν degrees of freedom, expected for coupling to ν independent channels.

Interpretation

There are additional channels!

The exact formulas (Sommers *et al.* 1999) posed tremendous numerical problems.

Therefore a phenomenological approach has been used for the line width distribution:

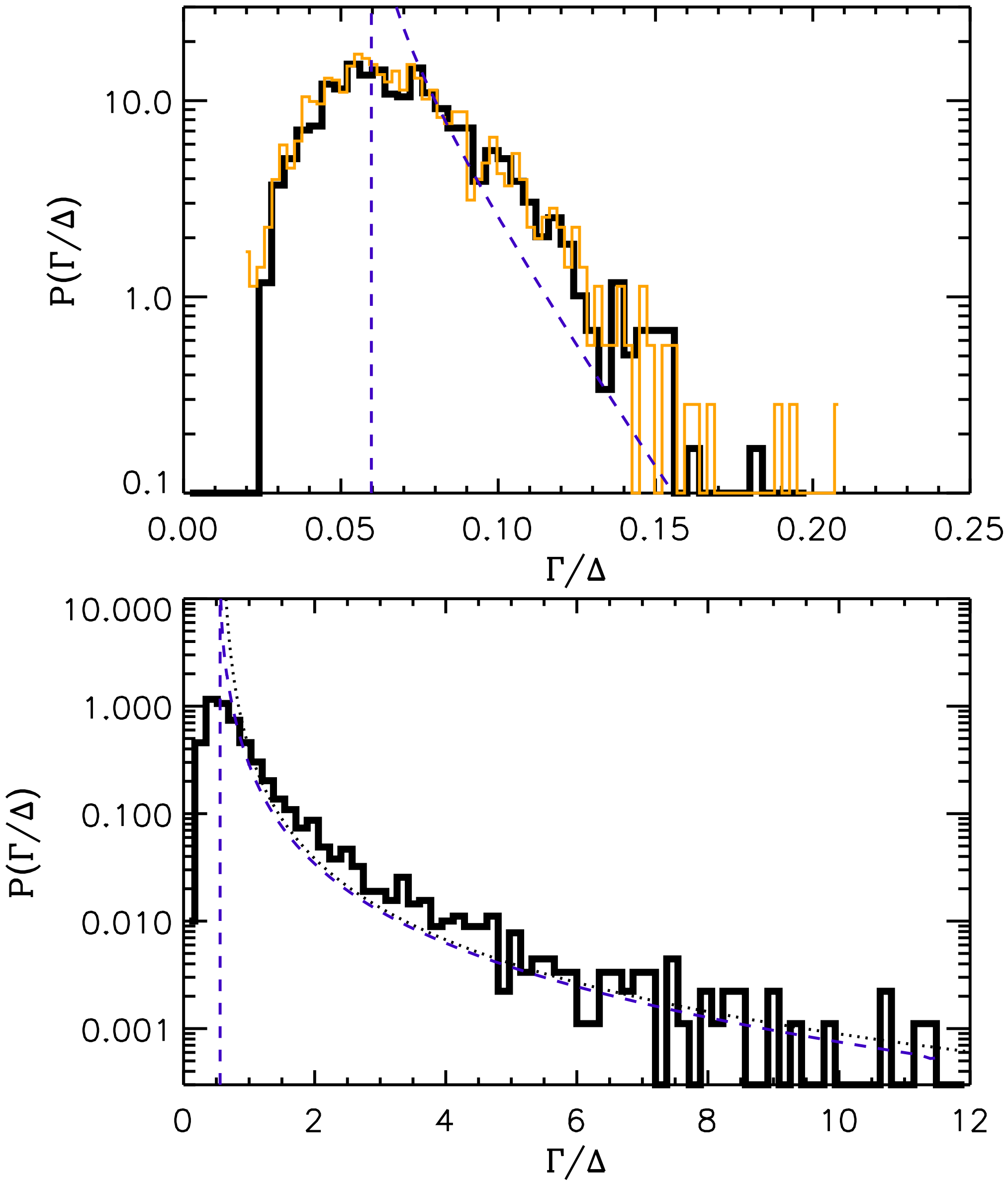
$$p(\Gamma) = \int \chi_\nu(\Gamma - \hat{\Gamma}) p_0(\Gamma) \, d\,\hat{\Gamma}$$

$p_0(\Gamma)$: one-channel distribution (antenna)

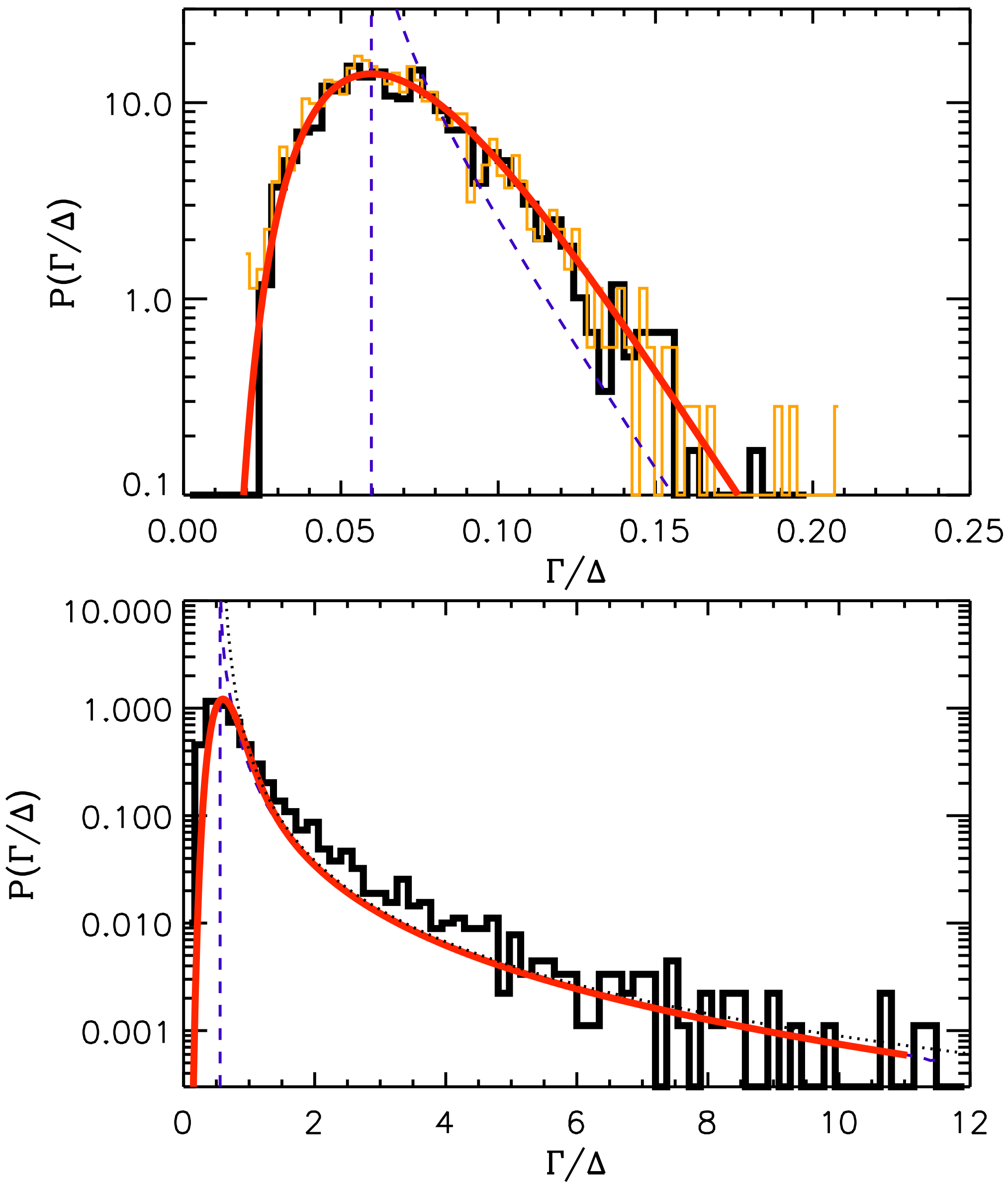
$\chi^2_\nu(\Gamma)$: chi-square distribution with ν degrees of freedom, expected for coupling to ν independent channels.

Exact in the non-overlapping regime, but questionable elsewhere!

Experimental results (*cont.*)

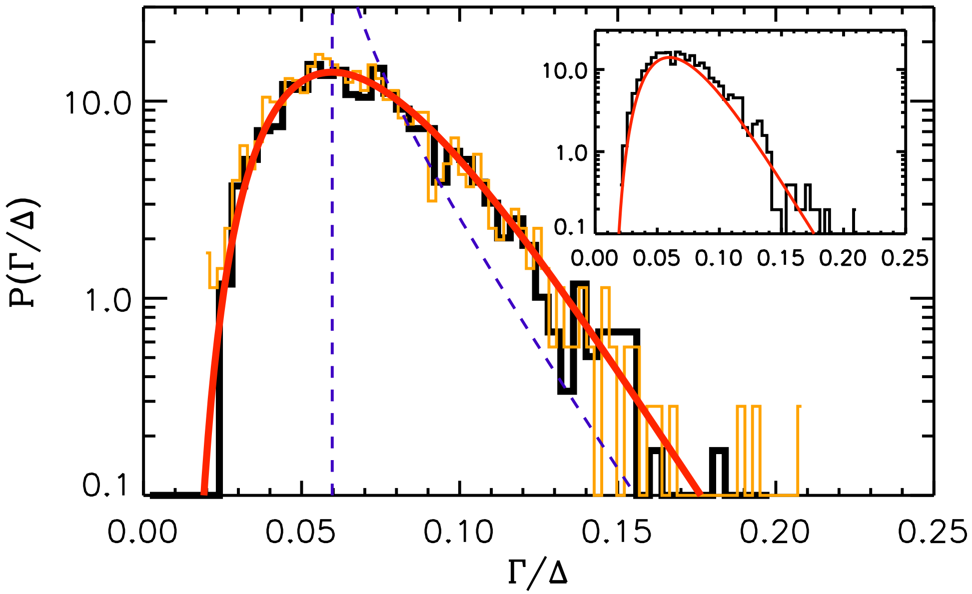


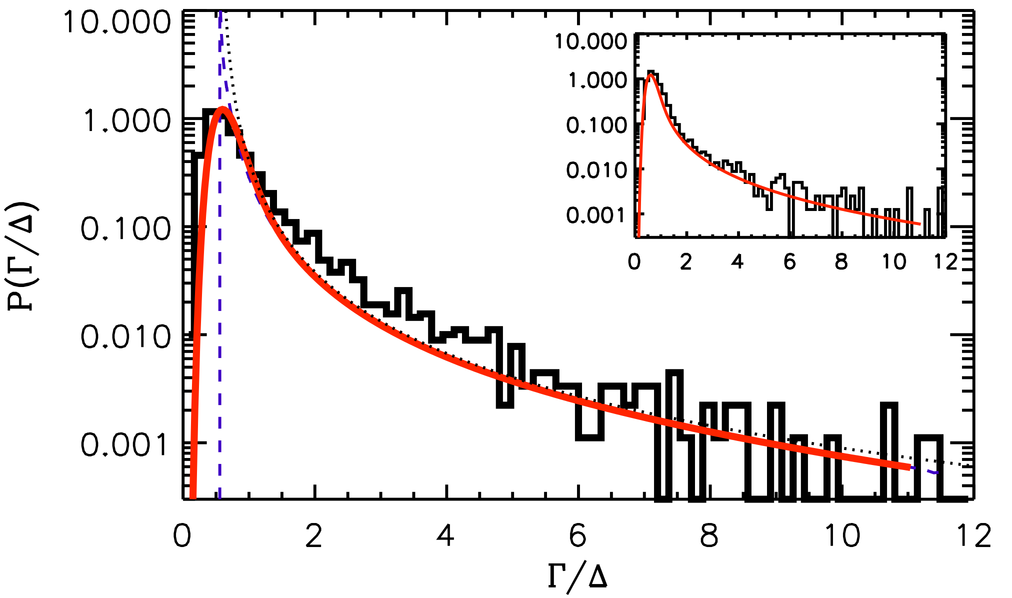
Experimental results (*cont.*)



Perfect agreement, assuming N=10 (top) and N=20 (bottom) weakly coupled additional channels.

Experimental results (*cont.*)





Perfect agreement, assuming N=10 (top) and N=20 (bottom) weakly coupled additional channels.

Insert: Simulation using the same parameters as in the experiment.

Folding approximation works very well!

Summary

- Harmonic inversion has passed the experimental test successfully!
- Resonances resolved in a regime where the line width exceeds the mean level spacings by a factor of 10 (in preliminary experiments even factors of 50 have been achieved!)
- Allows studies of hitherto unaccessible questions, such as
  - pole distance distributions
  - spectra level dynamics in the complex plane
  - fractal Weyl law
  - . . .

Thanks!

Coworkers:

U. Kuhl
R. Höhmann

Cooperations:

R. Mendez, Cuernavaca, Mexico
J. Main, Stuttgart

The experiments have been supported by the DFG via the

FG 760 “Scattering Systems with Complex Dynamics”.