Spreading of Wavepackets in Disordered Nonlinear Systems

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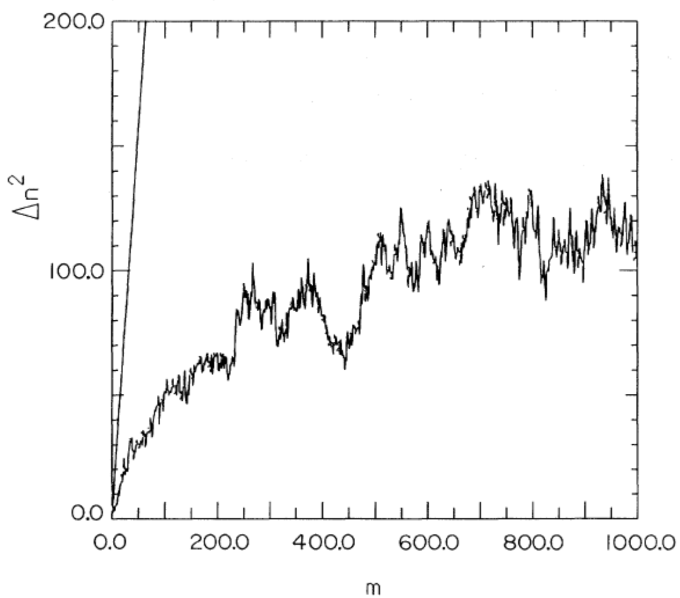
- defining the problem
- on spatial and temporal scales
- numerical observations
- a selftrapping theorem
- universal subdiffusive spreading

- G. Kopidakis, S. Komineas, SF, S. Aubry PRL100 (2008) 084103
- SF, Ch. Skokos, D. Krimer arXiv:0805.4693 (2008)

Some history on another (related?) model:

- Chirikov (long time ago): standard map, diffusion in momentum space
- Fishman, Grempel, Prange (1982): localization in QKR
- Benvenuto, Casati, Pikovsky, Shepelyansky (1991): nonlinear QKR

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} - \beta|\psi|^2\psi + k(\cos x)\psi\sum_{m=-\infty}^{+\infty}\delta(t-mT)$$



Defining the problem

- a disordered medium
- linear equations of motion: all eigenstates are localized
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet

Will it delocalize or stay localized?

Nonlinearity and disorder – model I

$$\mathcal{H}_D = \sum_l \epsilon_l |\psi_l|^2 - \frac{\beta}{2}|\psi_l|^4 - (\psi_{l+1}\psi_l^\star + c.c.)$$

ϵ_l uniformly from $\left[-\frac{W}{2}, \frac{W}{2}\right]$ $\quad \dot{\psi}_l = \partial\mathcal{H}_D / \partial(i\psi_l^\star)$

Conserved quantities: energy and norm

Nonlinearity and disorder – model II

$$\mathcal{H}_K = \sum_l \frac{p_l^2}{2} + \frac{\tilde{\epsilon}_l}{2}u_l^2 + \frac{1}{4}u_l^4 + \frac{1}{2W}(u_{l+1} - u_l)^2$$

$$\ddot{u}_l = -\partial\mathcal{H}_K / \partial u_l$$

$\tilde{\epsilon}_l$ uniformly from $\left[\frac{1}{2}, \frac{3}{2}\right]$

The linear case: $\beta = 0$

Stationary states: $\lambda A_l = \epsilon_l A_l - (A_{l+1} + A_{l-1})$

Eigenvectors = normal modes $A_{\nu,l}$

amplitude / momentum $a_\nu \; \dot{a}_\nu$ **Position** $X_\nu = \sum_l l A_{\nu,l}^2$

Distribution characterization

Second moment: $m_2 = \sum_l (l - l_0)^2 z_l$

Participation number: $P = (\sum_l z_l)^2 / \sum_l z_l^2$

Model I: norm density distributions **Model II: energy density distributions**

We consider a wavepacket at $t = 0$ which is given by a single site excitation $\Psi_l = \delta_{l,l_0}$ with $\epsilon_{l_0} = 0$ for DNLS, and $x_l = X\delta_{l,l_0}$ with $p_l = 0$ and $\tilde{\epsilon}_{l_0} = 1$ for KG. The value of X controls the energy E in the latter case.

Spatial and temporal scales

W=4 :

Eigenvalue (frequency) spectrum width: $\Delta = W + 4$ **8**

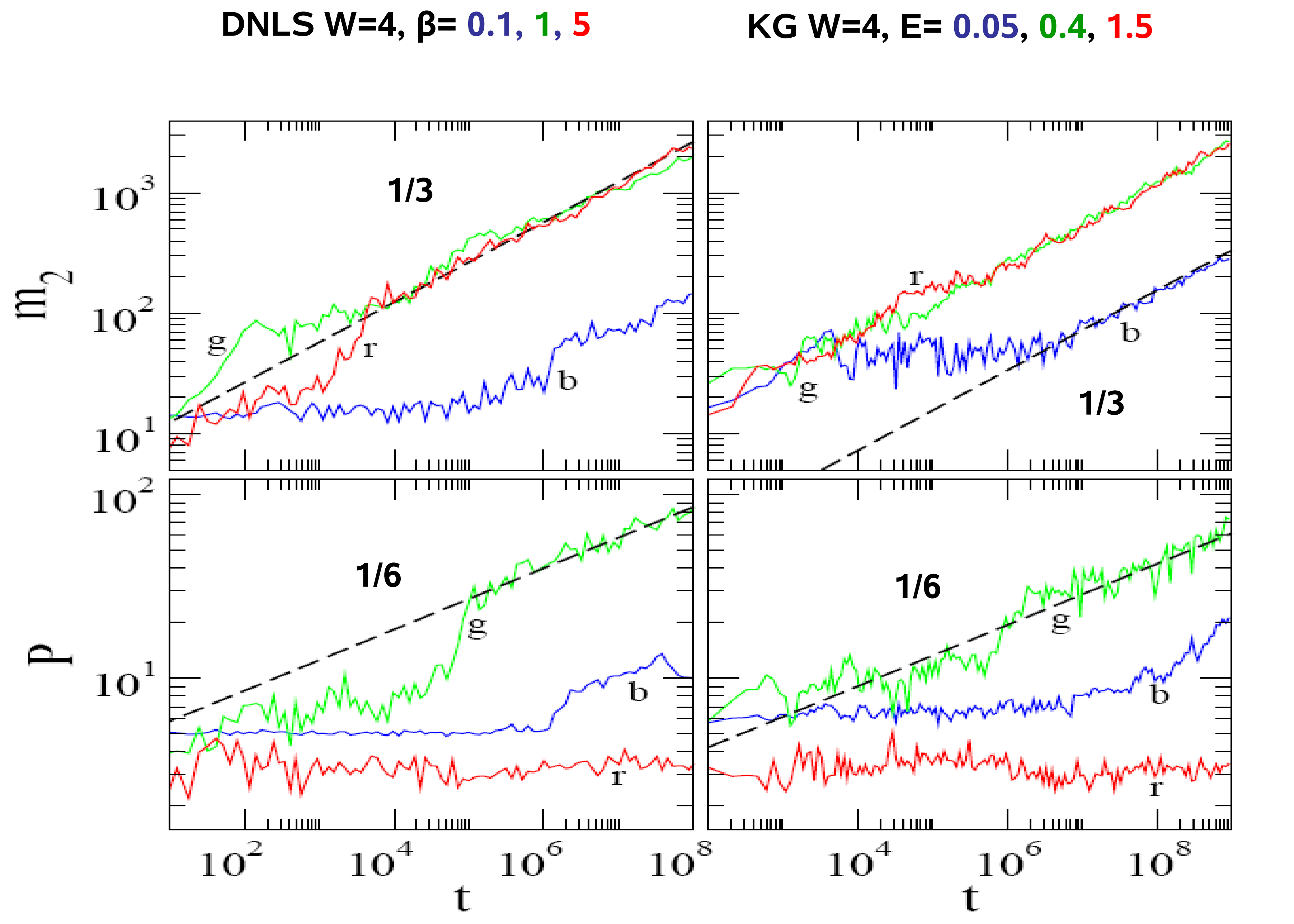
Localization volume of eigenstate: $P_\nu = 1/\sum_l A^4_{\nu,l}$ **18 (sites)**

Average frequency spacing inside localization volume: $\overline{\Delta\lambda} \approx \Delta/\overline{P_\nu}$ **0.43**

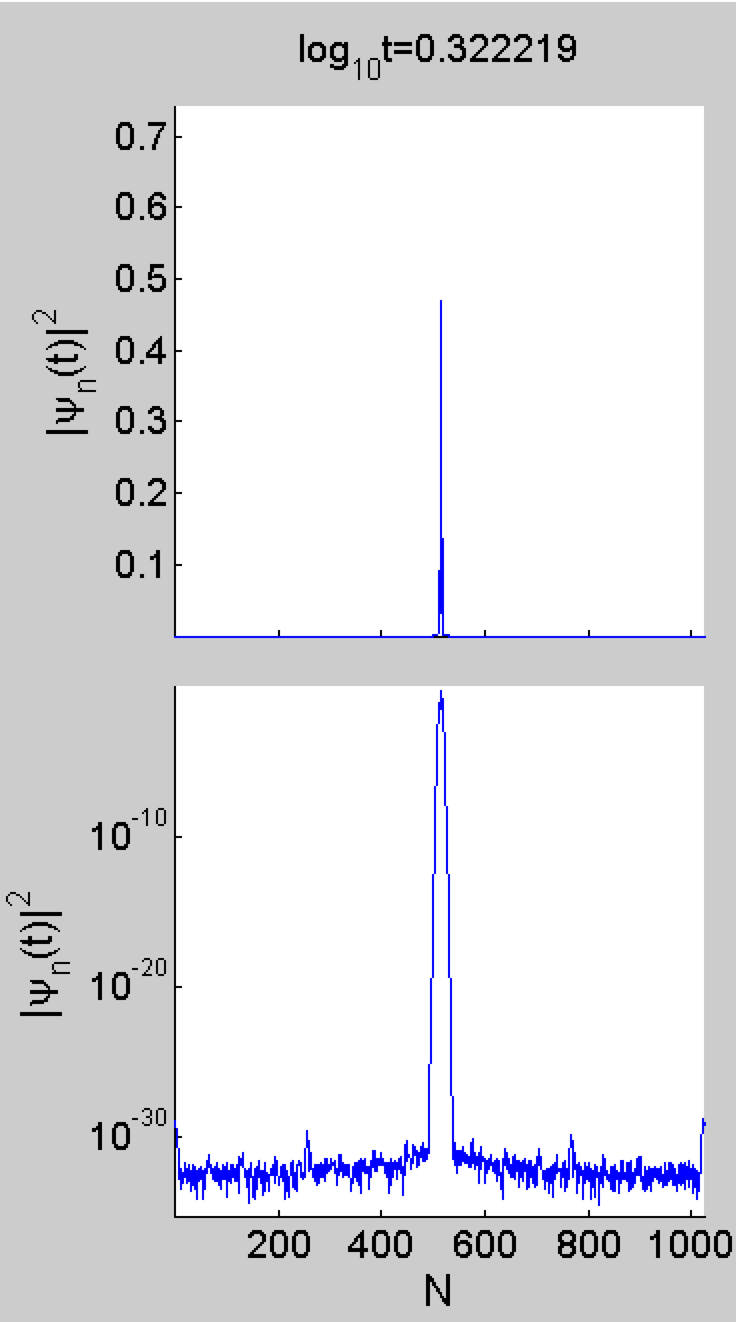
Nonlinearity induced frequency shift: $\beta|\psi|^2$

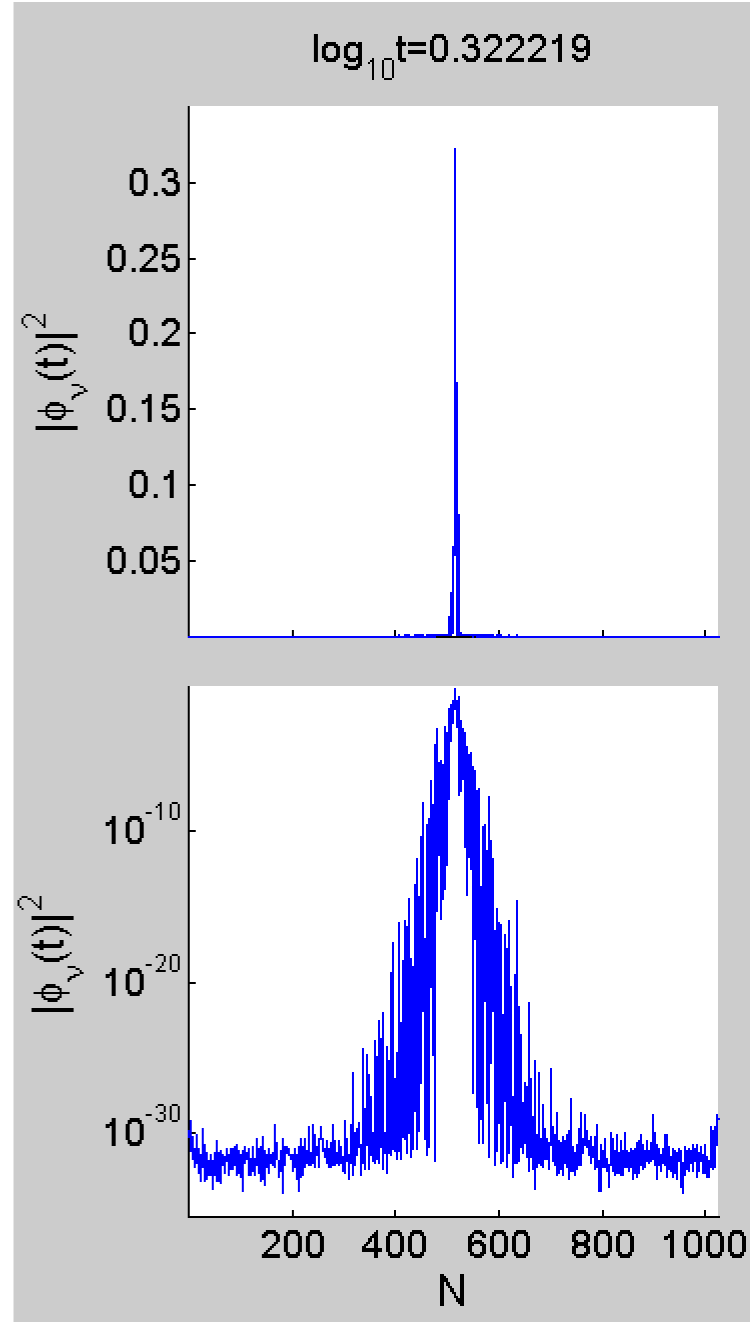
Equations in normal mode space:

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi_{\nu_1} \phi^*_{\nu_2} \phi_{\nu_3} \qquad I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

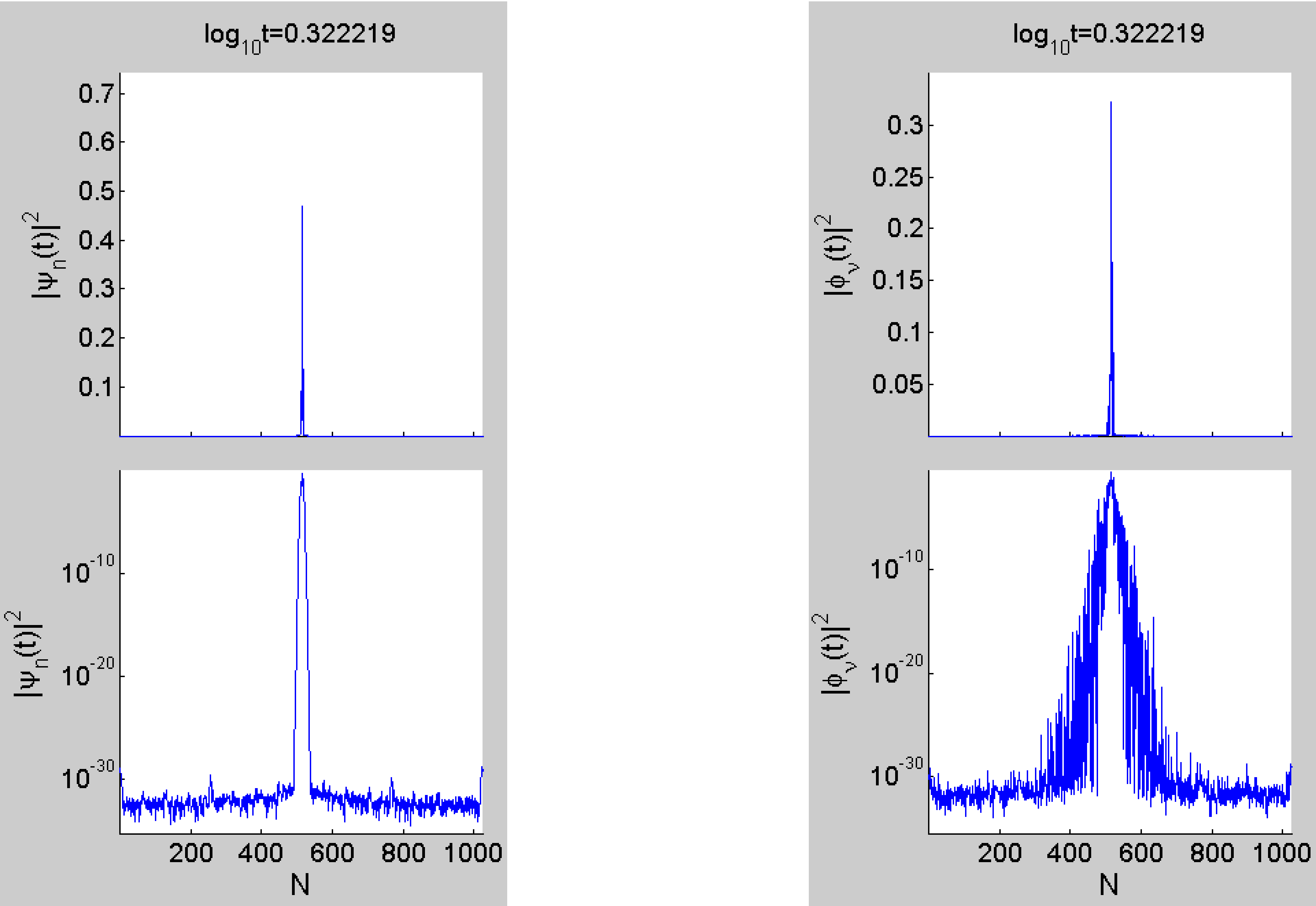

DNLS W=4, β= 0.1, 1, 5
KG W=4, E= 0.05, 0.4, 1.5
m2
P
t
1/3
1/3
1/6
1/6
g
r
b
10^1
10^2
10^3
10^4
10^6
10^8

Real space **DNLS W=4, β= 0.1** **Normal mode space**

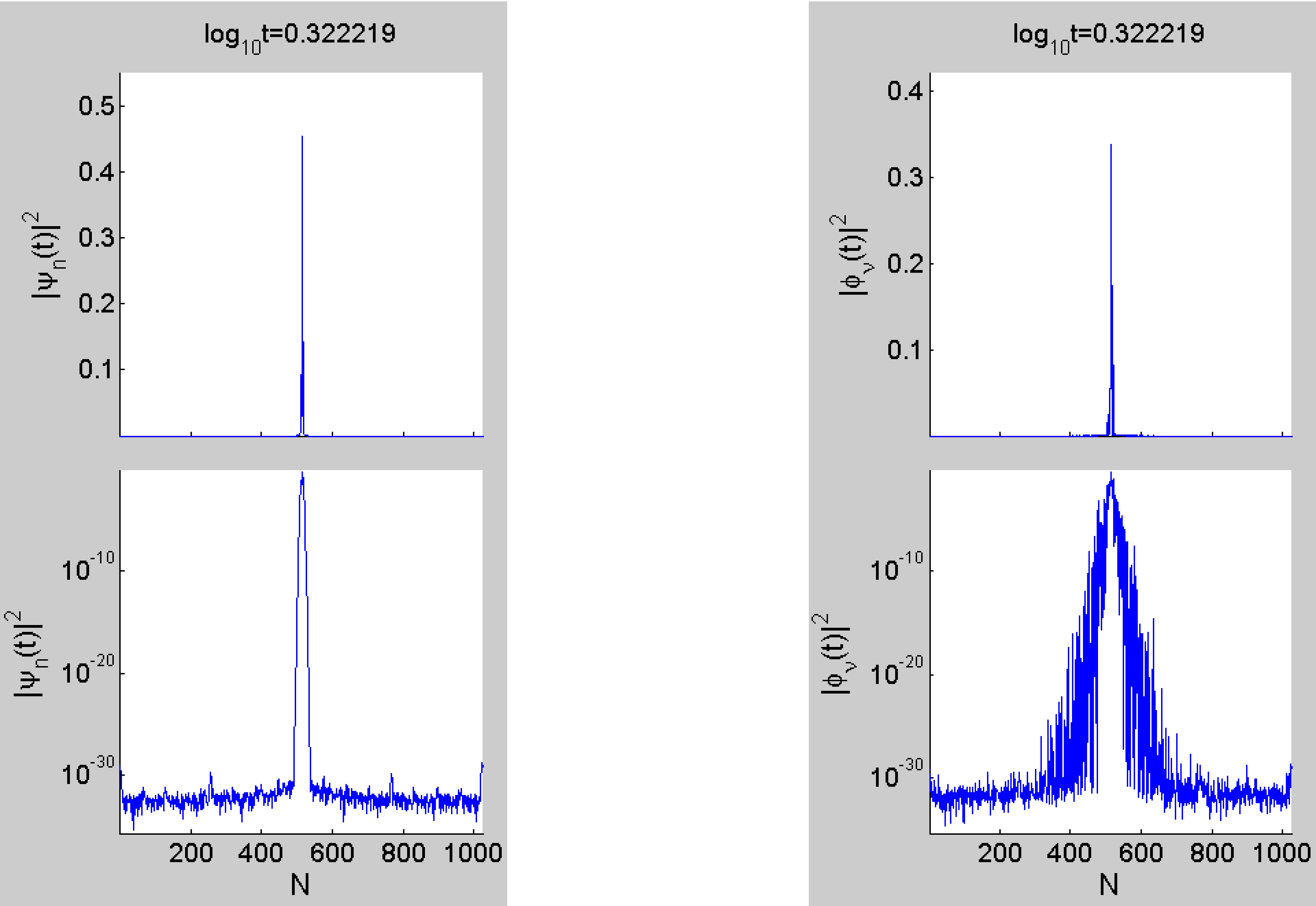




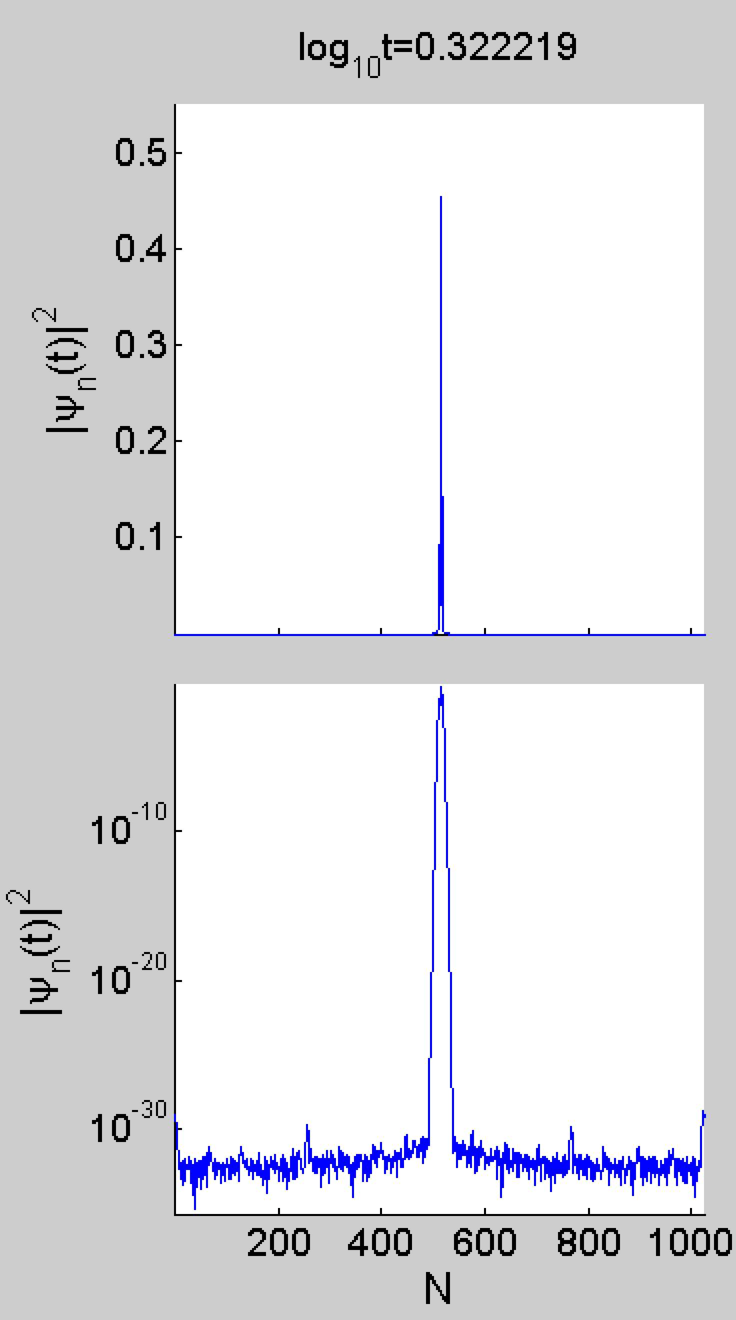
Real space **DNLS W=4, β= 0.1** **Normal mode space**

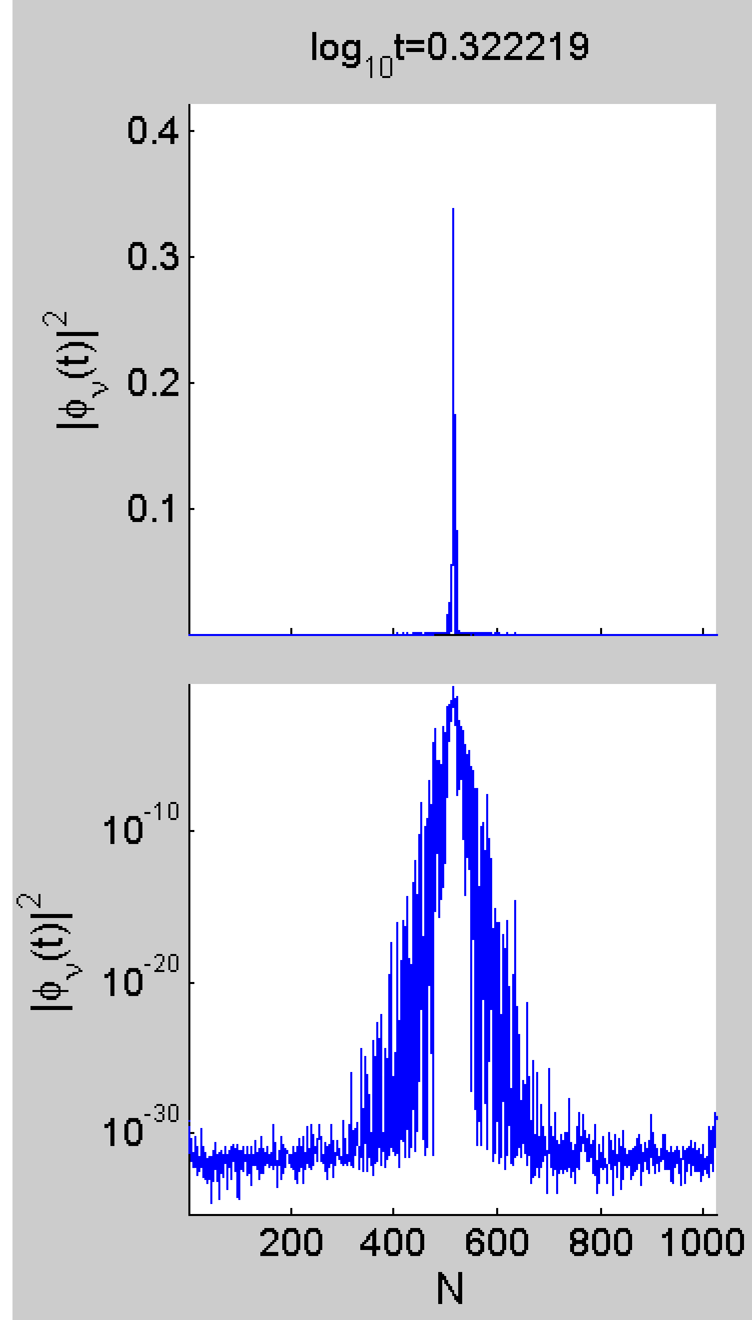


Real space **DNLS W=4, β= 0.3** **Normal mode space**

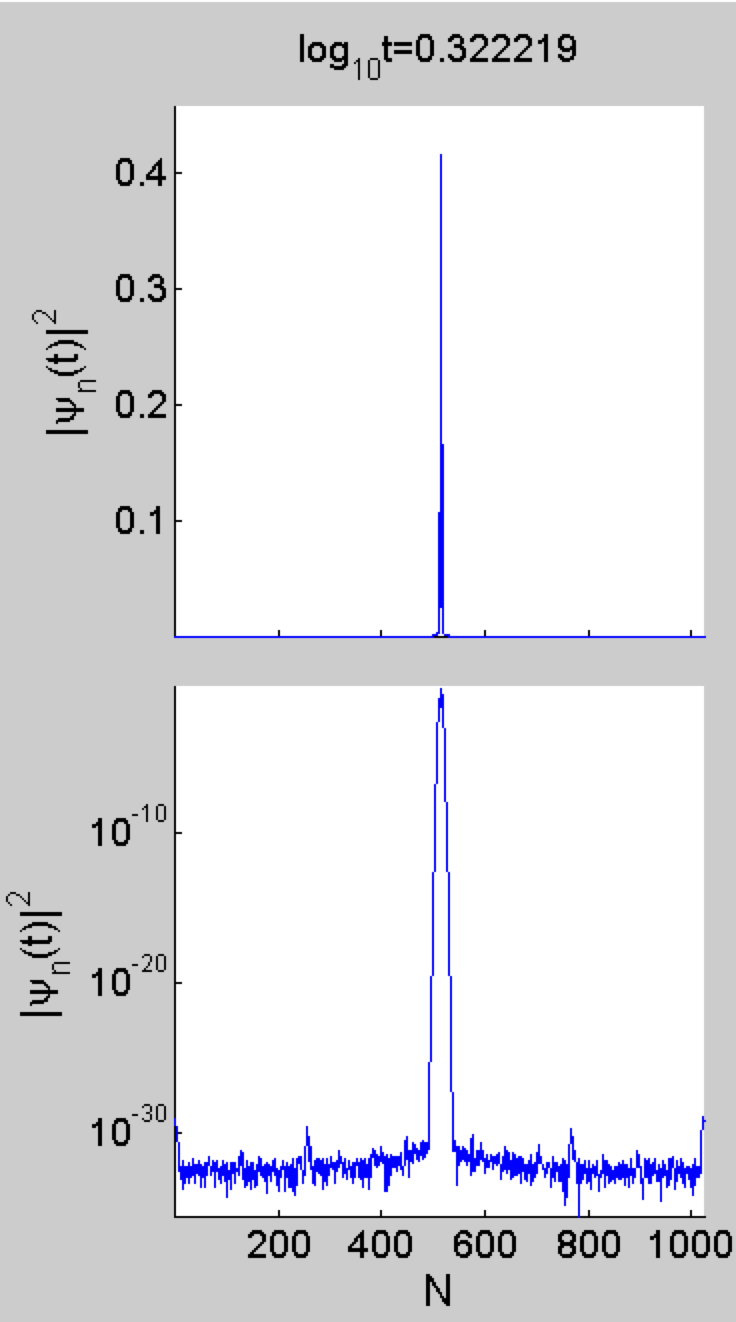


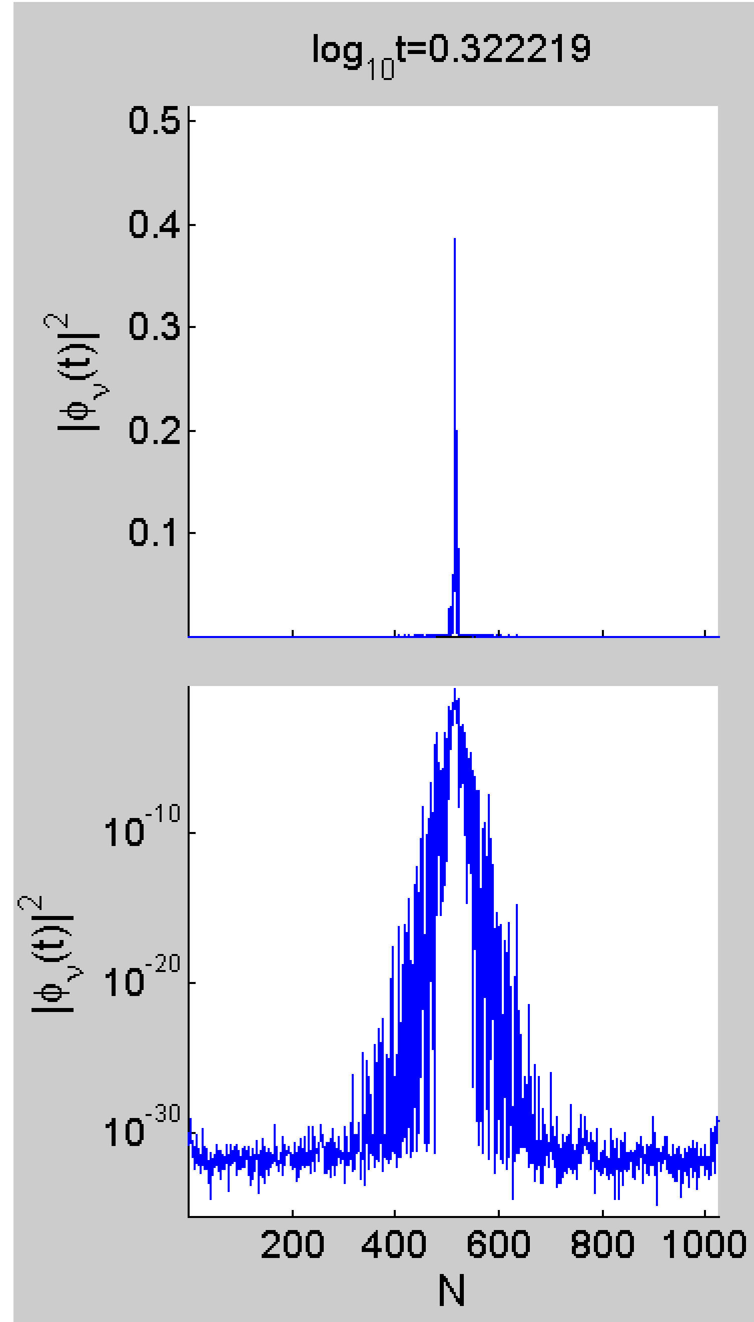
Real space **DNLS W=4, β= 0.3** **Normal mode space**



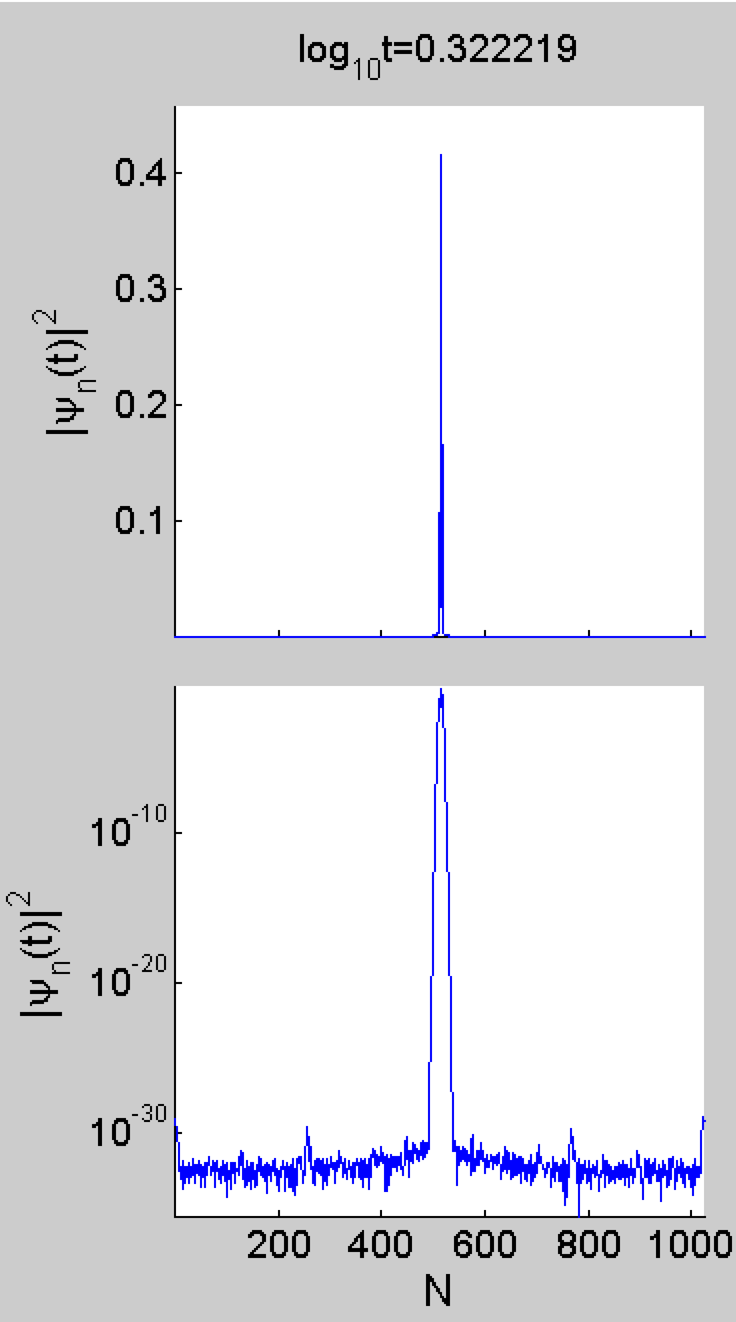


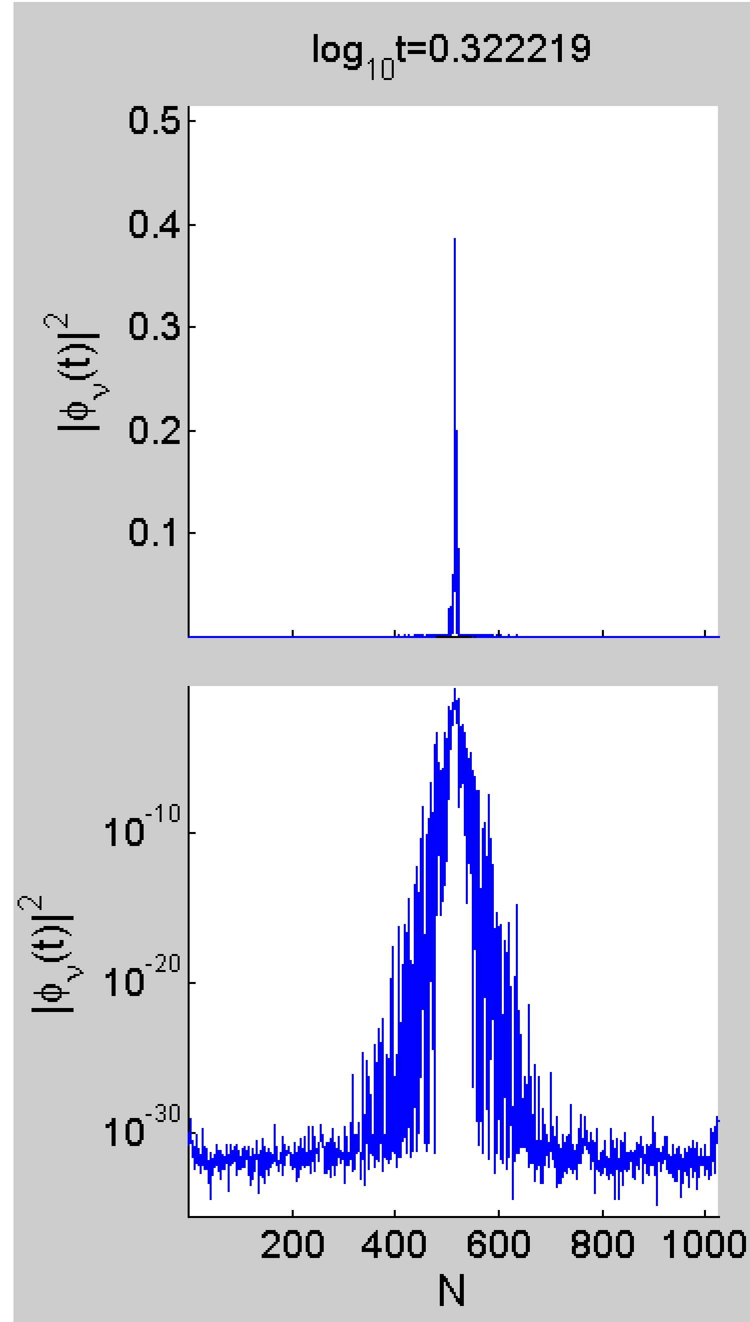
Real space **DNLS W=4, β= 1** **Normal mode space**





Real space **DNLS W=4, β= 1** **Normal mode space**

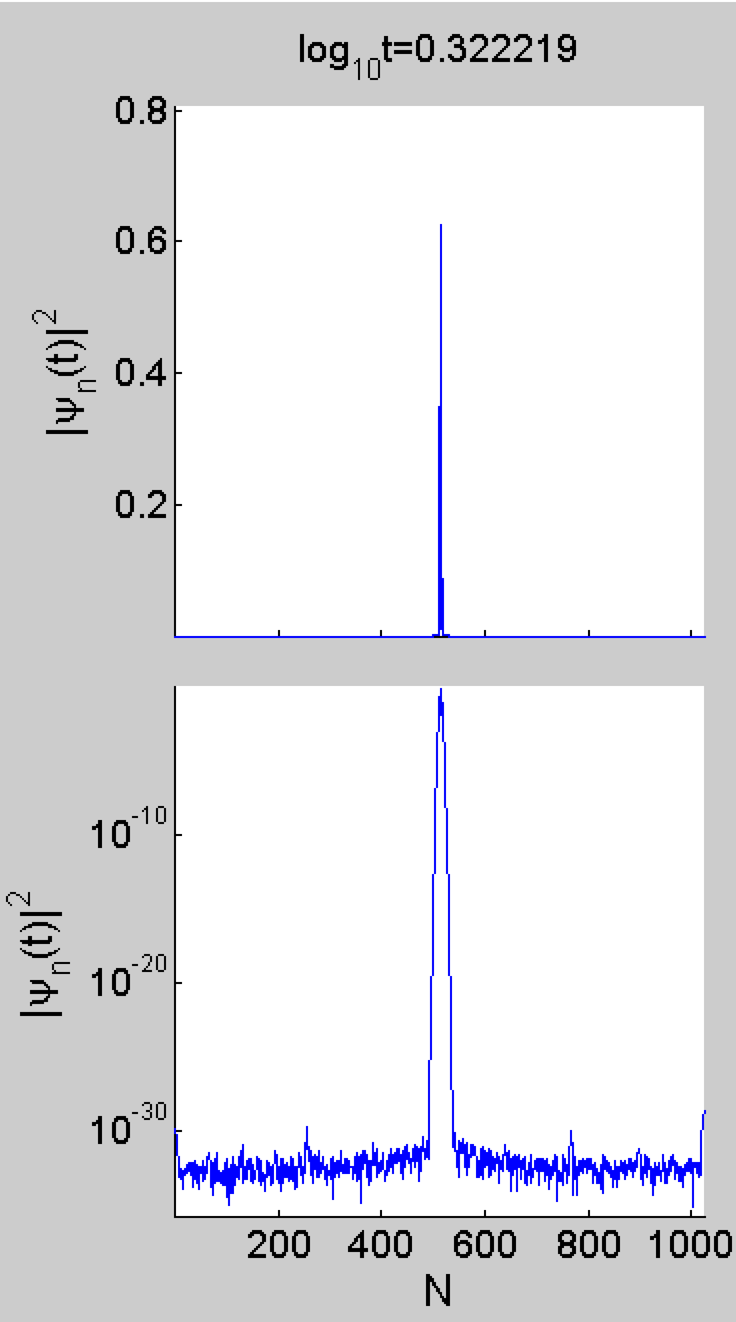


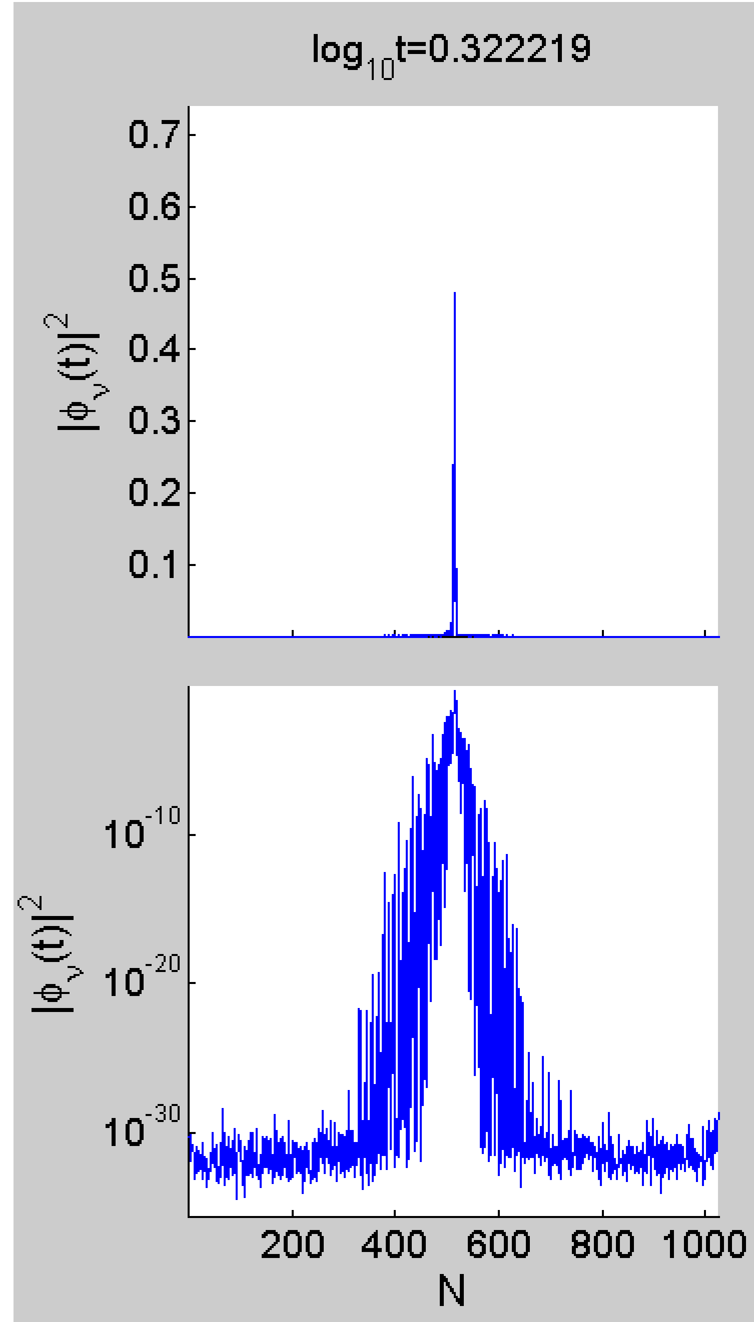


Real space

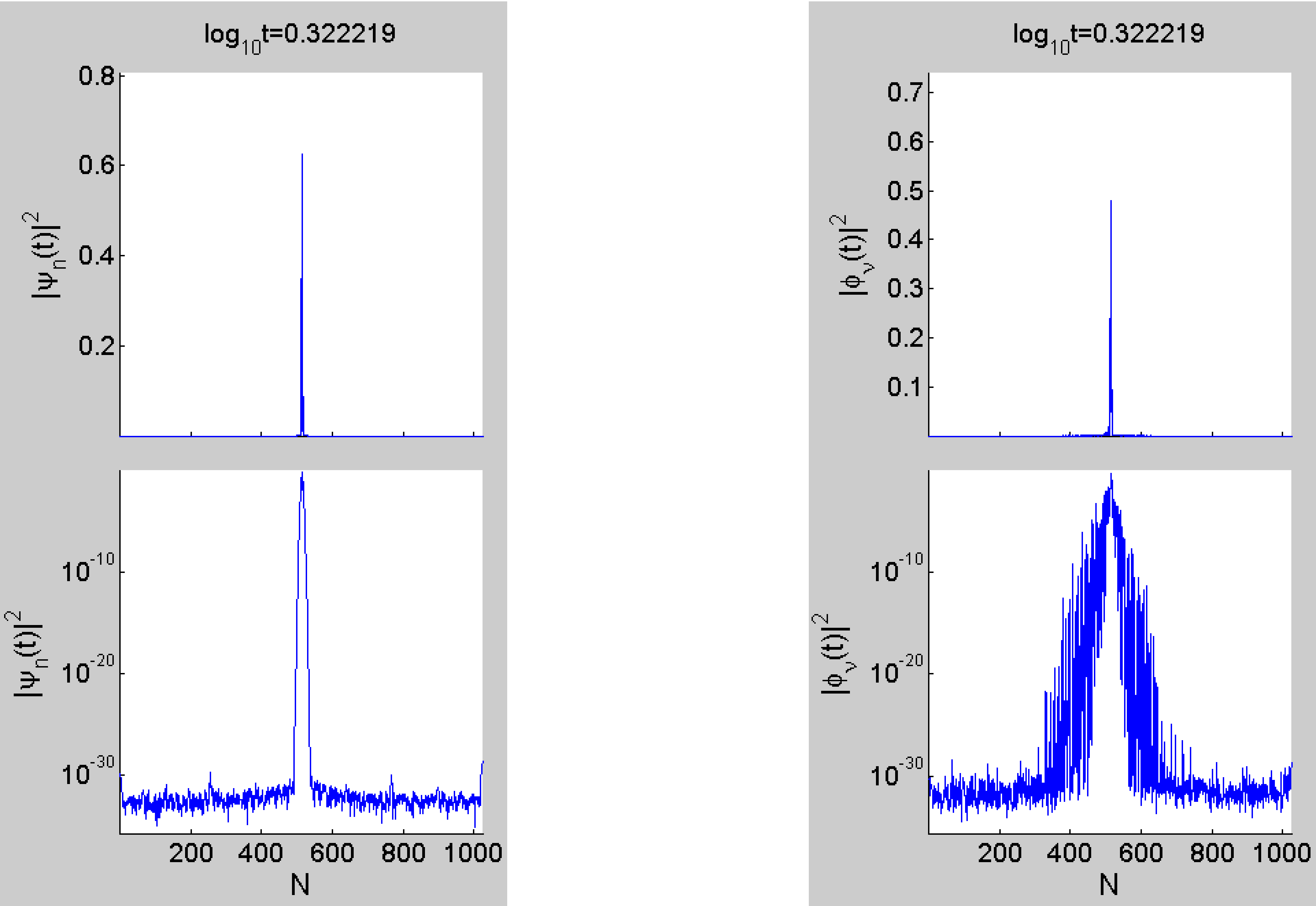
DNLS W=4, β= 5

Normal mode space

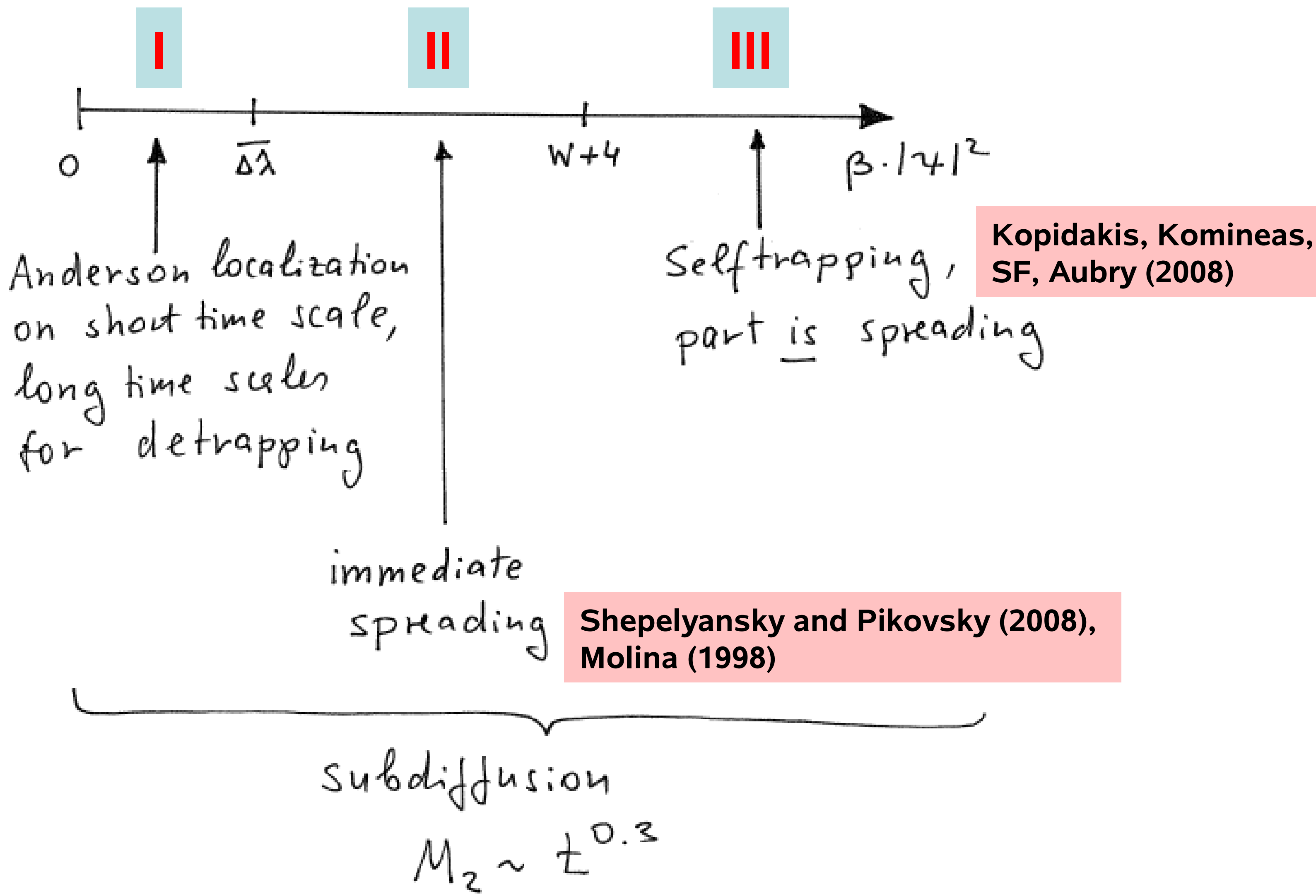




Real space **DNLS W=4, β= 5** **Normal mode space**



The emerging picture

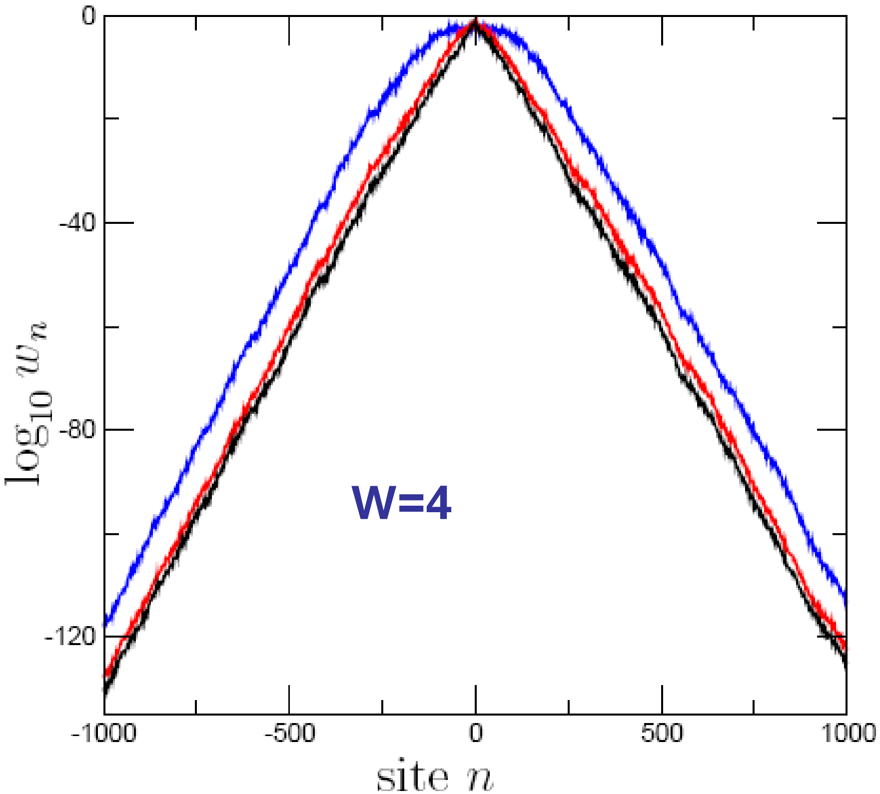


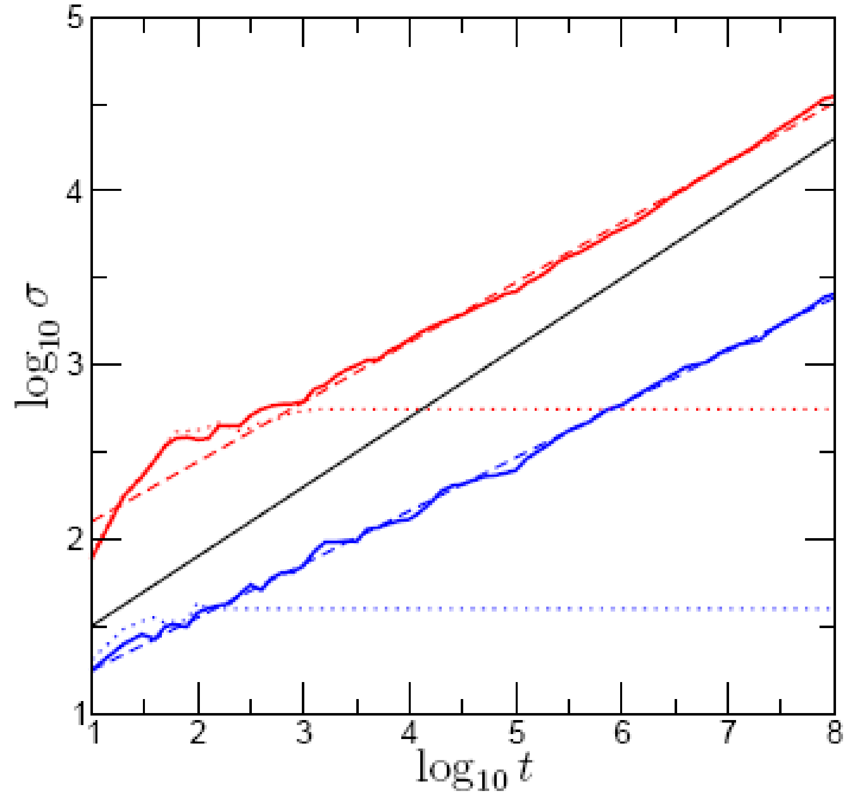
II

Immediate spreading : evolution of second moment

Shepelyansky PRL 1993, Molina PRB 1998,
Pikovsky and Shepelyansky PRL 100 (2008) 094101

Second moment grows subdiffusively, therefore complete delocalization





III

A theorem for selftrapping

G. Kopidakis, S. Komineas, S. Flach, S. Aubry
Phys. Rev. Lett. 100 (2008) 084103

plitudes. The consequence is that a part of the initial energy must remain well-focused at all times.

This proof is inspired by [11]. We split the total energy $\mathcal{H}_D = \langle\psi|\mathbf{L}|\psi\rangle + H_{NL}$ into the sum of its quadratic term of order 2 and its nonlinear terms of order strictly higher than 2. Then, $\mathbf{L}$ is a linear operator which is bounded from below (and above). In our specific example, we have $\langle\psi|\mathbf{L}|\psi\rangle \geq \omega_m\langle\psi|\psi\rangle = \omega_m S$ where $\omega_m \geq -2 - \frac{W}{2}$ is the lowest eigenvalue of $\mathbf{L}$. Otherwise, the higher order nonlinear terms have to be strictly negative.

If we assume that the wavepacket amplitudes spread to zero at infinite time, we have $\lim_{t\to+\infty}(\sup_n|\psi_n|) = 0$. Then $\lim_{t\to+\infty}(\sum_n|\psi_n|^4) < \lim_{t\to+\infty}(\sup_n|\psi_n^2|)(\sum_n|\psi_n|^2) = 0$ since $S = \sum_n \psi_n^2$ is time invariant. Consequently , for $t \to +\infty$ we have $\mathcal{H}_{NL} = 0$ and $\mathcal{H}_D \geq \omega_m \sum_n |\psi_n|^2 = \omega_m S$. Since $\mathcal{H}_D$ and S are both time invariant, this inequality should be fulfilled at all times. However when the initial amplitude A of the wavepacket is large enough, it cannot be initially fulfilled because the nonlinear energy diverges as $-A^4$ while the total norm diverges as A^2 only. For example, a wavepacket initially at a site 0 ($\psi_n = 0$ for $n \neq 0$ and $\psi_0 = \sqrt{A}$) has energy $\mathcal{H}_D = \epsilon_0 A^2 - \frac{1}{2}A^4$. Consequently, the above inequality is not fulfilled when $A^2 > -2(\omega_m - \epsilon_0) > 0$. Thus such an initial wavepacket cannot spread to zero amplitudes at infinite time.

Main idea: presence of two conserved quantities.
If norm is concentrated in a small volume, and if nonlinearity is large enough, wavepacket can not spread uniformly ad infinitum, since it can not convert all its anharmonic (interaction) energy part into the bounded kinetic energy

Therefore no complete delocalization for some $\beta > \beta_c$ **= W+4 …**

Explaining subdiffusion

- at some time t packet contains 1/n modes: $1/n \gg \overline{P_\nu}$
- each mode on average has norm $|\phi_\nu|^2 \sim n \ll 1$
- the second moment amounts to $m_2 \sim 1/n^2$

Two mechanisms of exciting a cold exterior mode:

- heated up by the packet (nonresonant process)
- directly excited by a packet mode (resonant process)
- in both cases the relevant modes are in a layer of the width of the localization volume at the edge of the packet

Heating

Simplest assumption:

- all modes in packet evolve chaotic
- all phases decohere sufficiently fast
- spreading = heating of cold exterior

exterior mode:

$$i\dot{\phi}_\mu \approx \lambda_\mu \phi_\mu + \beta n^{3/2} f(t)$$

$$\langle f(t) f(t') \rangle = \delta(t - t')$$

$$|\phi_\mu|^2(t) \sim \beta^2 n^3 t$$

The momentary diffusion rate of packet equals the inverse time the exterior mode needs to heat up to the packet level:

$$D = 1/T \sim \beta^2 n^2$$

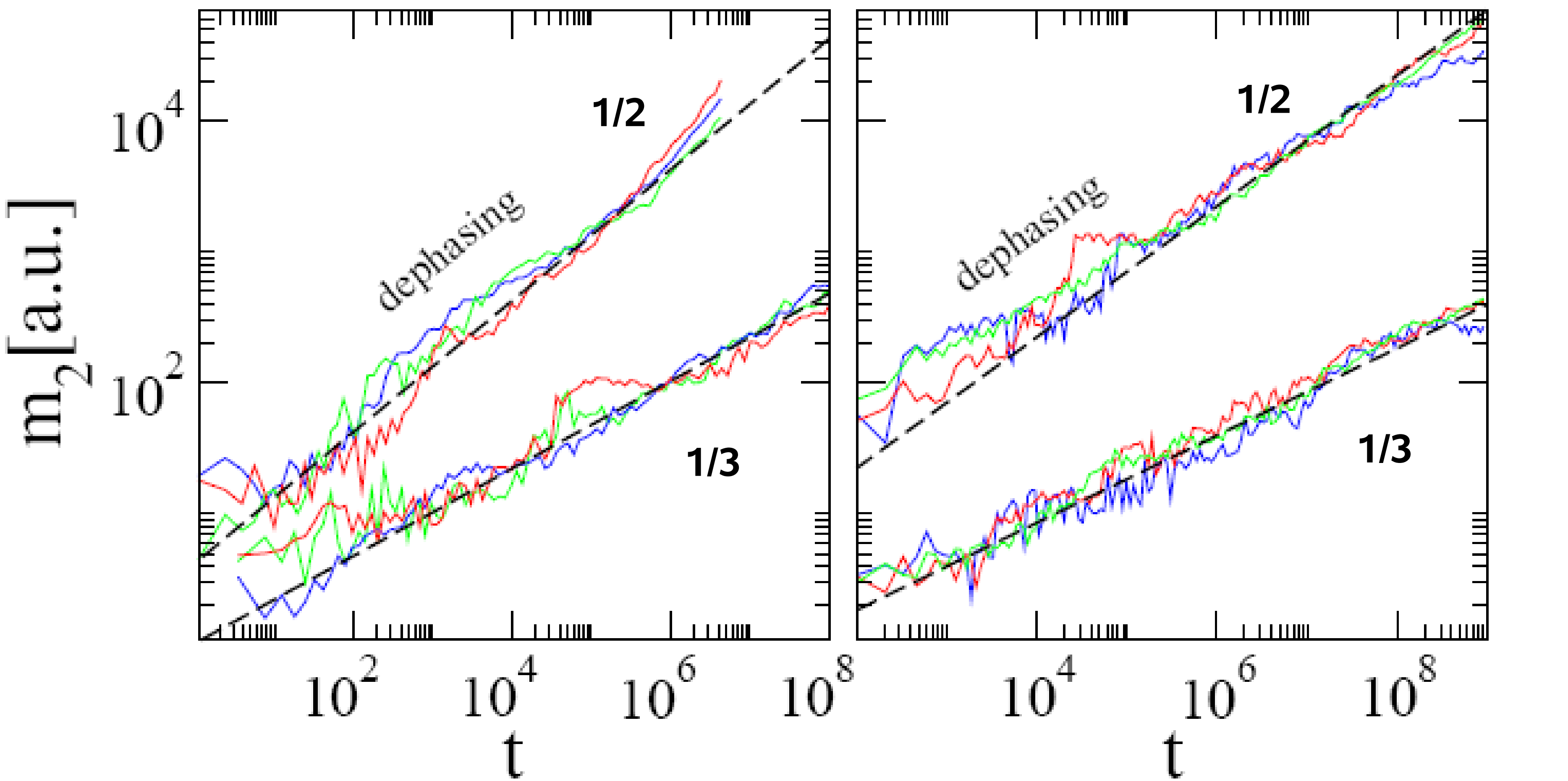
and therefore …

$$\boxed{m_2 \sim \beta t^{1/2}}$$

We test this prediction by additionally *dephasing the normal modes:*

DNLS
W=4, β=3
W=7, β=4
W=10, β=6

KG
W=10, E=0.25
W=7, E=0.3
W=4, E=0.4



Therefore:

- not all modes in packet evolve chaotic
- not all phases decohere sufficiently fast
- but spreading = heating of cold exterior?
- need to estimate the number of resonant modes

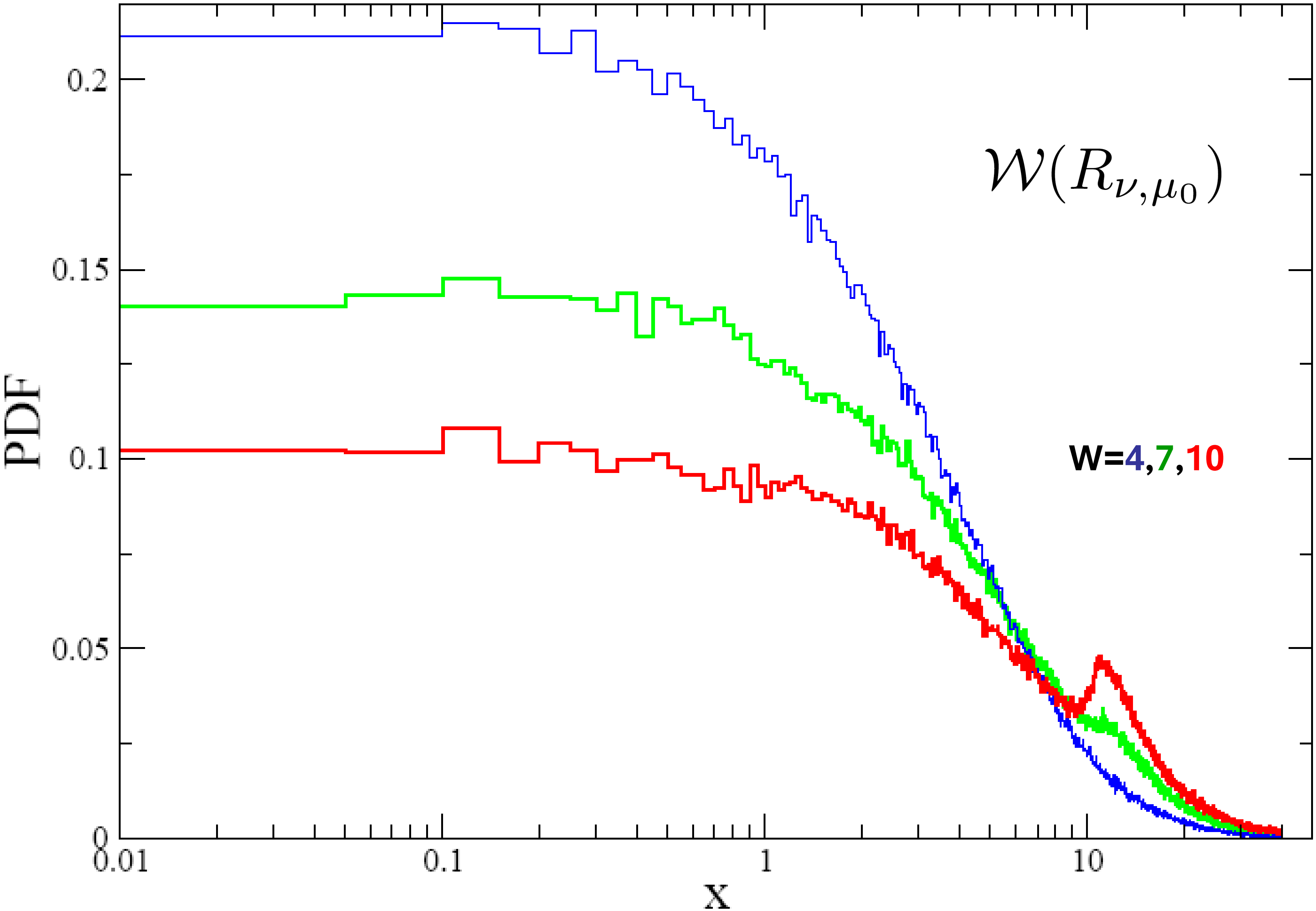
Perturbation approach: given a mode $|\phi_\nu|^2 = n$

$$|\phi_\mu| = \beta n R_{\nu,\mu}^{-1} |\phi_\nu| \ , \ R_{\nu,\mu} \sim \left| \frac{\lambda_\nu - \lambda_\mu}{I_{\mu,\nu,\nu,\nu}} \right|$$

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

Resonant interaction if $R_{\nu,\mu} < \beta n$

We compute $\mathcal{W}(R_{\nu,\mu_0})$ $\quad R_{\nu,\mu_0} = \min_{\mu\neq\nu} R_{\nu,\mu}$



$\mathcal{W}(R_{\nu,\mu_0})$
W=4,7,10
PDF
X
0.2
0.15
0.1
0.05
0
0.01
0.1
1
10

$$\mathcal{W}(x \to 0) \to const \neq 0$$

Probability for mode to be resonant is ~ βn

Number of resonant modes is constant on average

Fraction of resonant modes ~ βn

Heating: $$|\phi_\mu|^2(t) \sim \beta^4 n^5 t$$

$$D = 1/T \sim \beta^4 n^4$$

$$\boxed{m_2 \sim \beta^{4/3} t^{\alpha} \;,\; \alpha = 1/3}$$

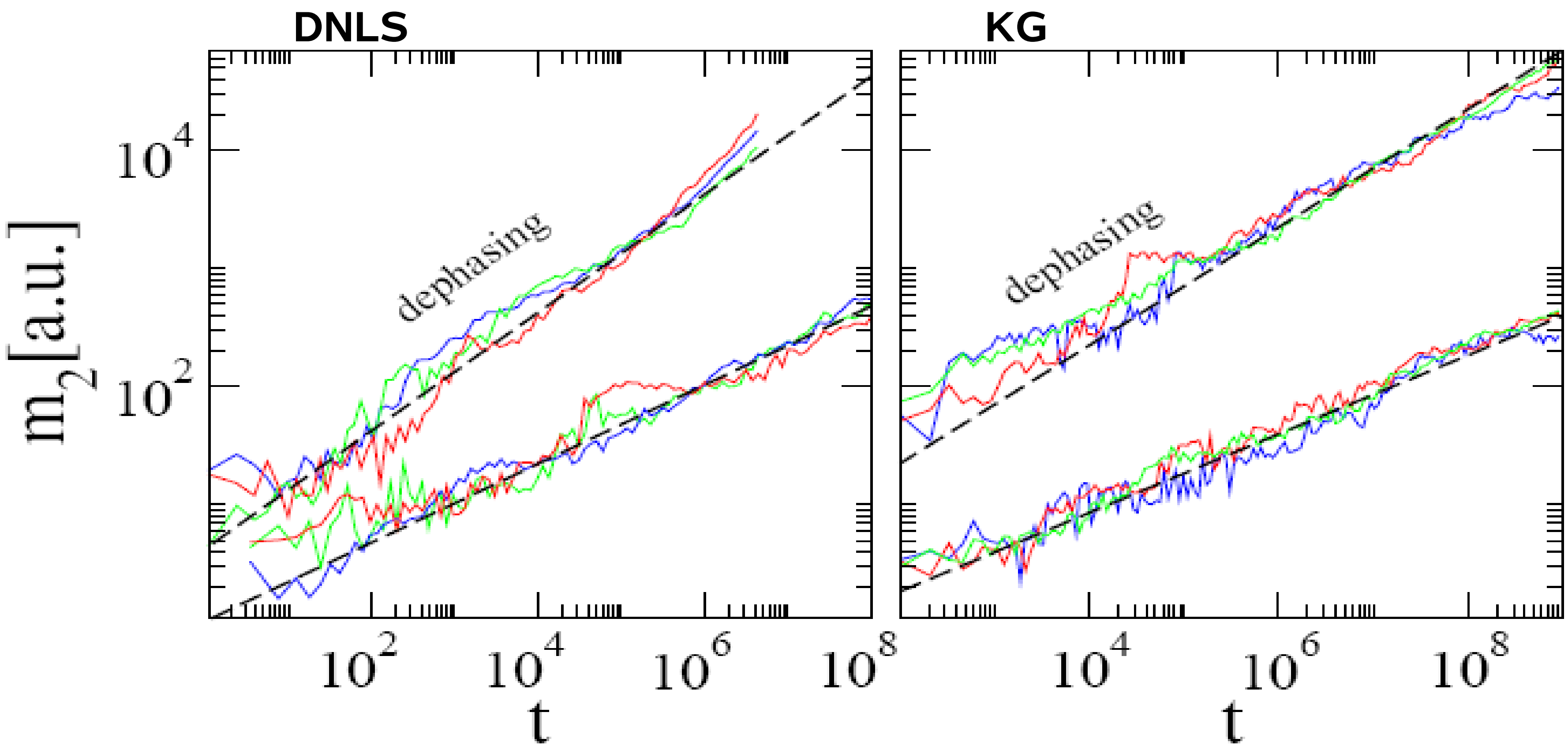
Direct resonant excitation: can be excluded for the same reasons!

We perform extensive computations

We averaged the measured exponent over 20 realizations:

$\alpha = 0.34 \pm 0.02$ (DNLS)

$\alpha = 0.34 \pm 0.05$ (KG)



Extensions to other dimensions and nonlinearities

$$i\dot{\psi}_{\boldsymbol{l}} = \epsilon_{\boldsymbol{l}}\psi_{\boldsymbol{l}} - \beta|\psi_{\boldsymbol{l}}|^{\sigma}\psi_{\boldsymbol{l}} - \sum_{\boldsymbol{m}\in D(\boldsymbol{l})}\psi_{\boldsymbol{m}} \qquad \sigma > 0$$

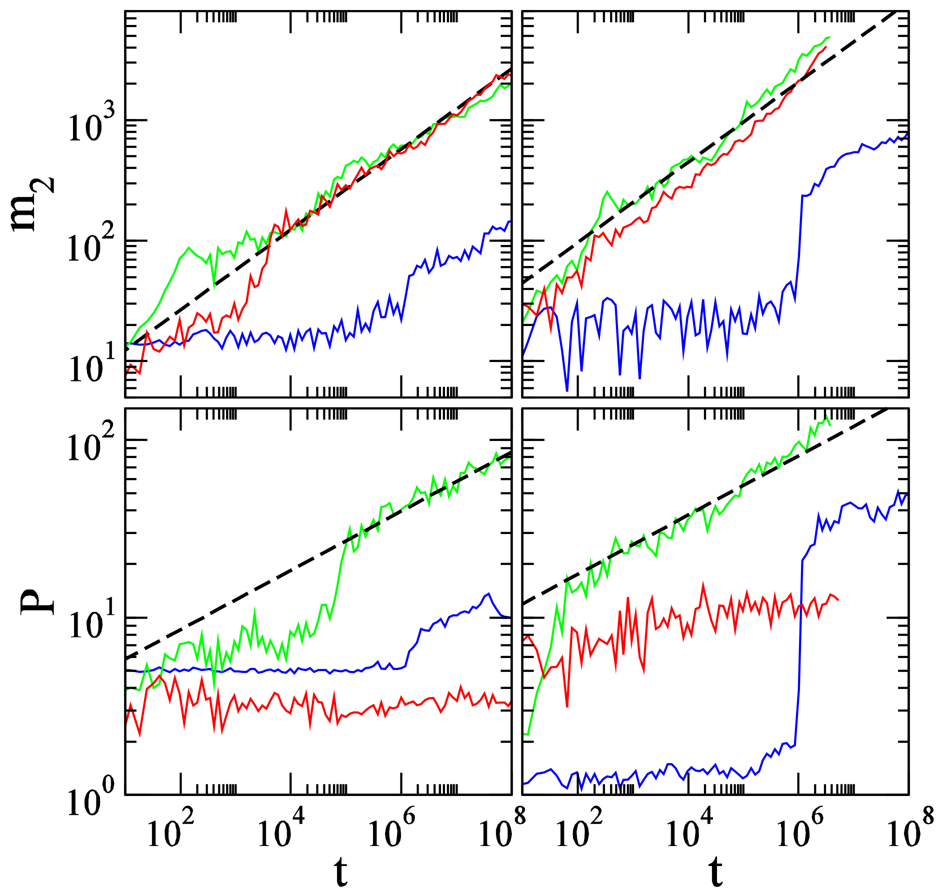
Dephasing of NMs yields $\tilde{\alpha} = 2/(2+d\sigma)$

If $\quad \mathcal{W}(x \to 0) \to const \neq 0$

$$m_2 \sim \left(\beta^4 t\right)^{\alpha}\ ,\ \alpha = \frac{2}{2+d(\sigma+2)}$$

The exponent α is bounded from above by $\alpha_{max} = 1/2$, which is obtained for $d = 1$ and $\sigma \to 0$. For the above studied two-body interaction $\sigma = 2$ we predict $\alpha(d = 2) = 1/5$ and $\alpha(d = 3) = 1/7$.

Single site initial excitation versus one NM excitation



Single site

One NM

Blue curves:
regime of weak nonlinearity

Green curves:
intermediate regime

Red curves:
selftrapping regime

Single site excitation [W=4; β=0.1 (blue), β=1 (green), β=5 (red)].

One NM excitation [W=4; β=0.6 (blue), β=5 (green), β=20 (red)).

Conclusions

• G. Kopidakis, S. Komineas, SF, S. Aubry PRL100 (2008) 084103
• SF, Ch. Skokos, D. Krimer arXiv:0805.4693 (2008)

• strong nonlinearity: partial localization due selftrapping (discrete breathers), but part of wavepacket delocalizes

• weak nonlinearity: Anderson localization on finite times: similar to FPU! After that – detrapping, and wavepacket delocalizes

• intermediate nonlinearity: wavepacket delocalizes without transients

• subdiffusive spreading due to a finite number of resonant chaotic modes

• second moment of wavepacket ~ $t^{1/3}$

• results do not depend on presence or absence of norm conservation

• spreading is universal due to nonintegrability

• exponent does not depend on strength of nonlinearity and disorder

• ANDERSON LOCALIZATION IS DESTROYED BY THE SLIGHTEST AMOUNT OF NONLINEARITY ! ... ?